Coxeter groups, the Davis complex, and isolated flats

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Abstract

Coxter groups arose as a natural generalization of reflection groups. J. Tits defined them in a simple way using generators and relations, that is, using a group presentation $W \cong \langle S \mid R \rangle$. Coxeter groups have a wide range of applications; for example, every Weyl group may be realized as a finite, irreducible Coxeter group.

The **Davis complex** Σ is a geometric realization of Coxeter groups, which is CAT(0) for every Coxeter group. It has therefore been one of the first classes of examples for CAT(0) spaces. We first provide a general introduction to Coxeter groups and the Davis complex, and continue discussing when the Davis complex has so called *flats*.

Flats are convex subsets which are isometric to $\mathbb{R}^n$. We say that Σ has *isolated flats* if there exists a collection $\mathfrak{F}$ of flats in Σ, satisfyingthe *isolated* property:

(A) There is a constant $D < \infty$ such that each flat F of Σ lies in a tubular D-neighborhood of some $C \in \mathfrak{F}$.

(B) For each positive $r < \infty$, there is a constant $\rho = \rho(r) < \infty$ so that for any two distincit elements $C, C' \in \mathfrak{F}$ we have $\operatorname{diam}(\mathcal{N}_r(C) \cap \mathcal{N}_r(C')) < \rho$, where $\mathcal{N}_r(C)$ denotes the tubular r-neighborhood of C.

Given a Coxeter group and a set of generators S, we can read from the *Coxeter diagram* if the resulting Davis complex has isolated flats. This classification is due to work by P. Caprace. We introduce the necessary concepts and give examples of Coxeter groups where Σ has isolated flats.

Contents

1 Coxeter groups

Coxeter groups can be realized in a straightforward combinatorial fashion, wherin the presentation of the group is of central importance, and analysis for the group is frequently performed with little reference to the group's geometric structure. The combinatorial viewpoint will facilitate the proofs of many results.

Note. For a detailed treatment on free groups and group presentations, see [Elements, Loeh].

Definition 1.1. Let S be a set, $F(S)$ the free group over S, and $R = (r_j)_{j \in J}$ a family of words in $F(S)$. A **group presentation** is then defined by $\langle S \mid R \rangle := F(S)/\langle\langle R \rangle\rangle$, where $\langle\langle R \rangle\rangle$ is the smallest normal subgroup containing R.

A group G is **finitely presented** if there exists a finite generating set S and a finite set $R \subset F(S)$ of relators such that $G \cong \langle S \mid R \rangle$. A classical example of a finitely presented group is given by the following:

Example 1.2 ([Massey, §5.3])**.** The fundamental group of an orientable surface Σ_n with genus n (i.e. the connected sum of n tori) is given by the finite presentation:

$$\pi_1(\Sigma_n) \cong \left\langle a_1, b_1, \ldots, a_n, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} \right\rangle .$$

Definition 1.3 ([Davis, Definition 3.3.2])**.** Let I be an indexing set, and let $S = \{s_i\}_{i \in I}$. Let $M = (m_{ij})_{i,j \in I}$ be a matrix such that

- $m_{ii} = 1$ for all $i \in I$;
- $m_{ij} = m_{ji}$ for all $i, j \in I$; and
- $m_{ij} \in \{2, 3, 4, \ldots\} \cup \{\infty\}$ for all distinct $i, j \in I$.

Then M is called a **Coxeter matrix**. The **Coxeter group** $W = W_M$ associated to a Coxeter matrix M is the finite presentation:

$$W \cong \left\langle S \mid s_i^2 = \mathbb{1} \ \forall i \in I, \text{ and } (s_i s_j)^{m_{ij}} = \mathbb{1} \ \forall i \neq j \right\rangle .$$

The pair (W, S) is a **Coxeter system** and S is the set of **Coxeter generators**. The cardinality of S is called the **rank** of (W, S). If for $i \neq j$ we have $m_{ij} \in \{2, \infty\}$, then (W, S) is called **right-angled**.

Remark 1.4.

- Throughout our discussion, we will assume that the indexing set I is *finite*. For example, we wish that the group W acts cocompactly on a certain geometric realisation (the *Davis complex*). This is only the case when W is finitely generated (that is, when I is finite.)

- There is a one-to-one correspondence between Coxeter *matrices* and Coxeter *systems*, as demonstrated by the following proposition. It is proved using a faithful linear representation $\sigma : W \to GL_n(\mathbb{R})$, the *Tits representation.* (See section 2.)

Proposition 1.5. *Suppose M is a Coxeter matrix, and W the group with generating set S defined by the presentation associated to M.*

1. *For each $i \in I$, the element s_i is an involution.*
2. *Each s_i is a distinct group element in W.*
3. *$s_i s_j$ has order m_{ij}.*

An element of order 2 is also called an **involution**. The next lemma shows, directly from the presentation, that each $s \in S$ is an involution in the group W. The other properties may be shown with a *faithful* linear representation $\sigma : W \hookrightarrow GL_n(\mathbb{R})$, where $\sigma(s)^2 = \mathrm{id}$ and $(\sigma(s)\sigma(t))^{m_{st}} = \mathrm{id}$.

Lemma 1.6. *Let (W,S) be a Coxeter system. There is an epimorphism $\varepsilon : W \to \mathbb{Z}/2\mathbb{Z}$ induced by $\varepsilon(s) = -1$ for all $s \in S$.*

Proof. By definition, $W \cong F(S)/R$. By the universal property of free groups, we have a unique homomorphism

$$F(S) \to \mathbb{Z}/2$$

extending ε. This homomorphism factors through W, because $\varepsilon\left((s_i s_j)^{m_{ij}}\right) = ((-1)(-1))^{m_{ij}} = 1$.

$$\begin{tikzcd} S \arrow[r, "\varepsilon"] \arrow[d, hook] & \mathbb{Z}/2\mathbb{Z} \\ F \arrow[ur, dashed, "\exists!"] & \end{tikzcd}, \qquad \begin{tikzcd} F(S) \arrow[r, "\bar{\varepsilon}"] \arrow[d, "\pi"'] & \mathbb{Z}/2\mathbb{Z} \\ F(S)/R \arrow[ur, dashed] & \end{tikzcd}$$

□

Remark 1.7 ([Abramenko, Exercise 2.55]). Let M be a matrix as in Definition 1.3, but which is *not* symmetric. Then there are elements $s, t \in S$ such that the order of the image of st in (W, S) is not $m(s,t)$.

Note that with an appropriately chosen generator set S' and Coxeter matrix M', (W, S') is still a Coxeter system.

Proof. Let M be non-symmetric. Then there are some $s, t \in S$ such that $m_{st} \neq m_{ts}$, that is $(st)^n = (ts)^m = 1$ with $m \neq n$. Without loss of generality, let $m < n$. Assume by contradiction that $(st)^n = 1$ with $(st)^i \neq 1$ for all $i \in \{1, \dots, n-1\}$. There then holds

$$\begin{aligned} & st \cdots st && = 1 \\ \Leftrightarrow\ & s(ts \cdots ts)^{n-1} t && = 1 \\ \Leftrightarrow\ & s\,\underbrace{(ts)^m}_{=1}(ts)^{n-1-m} t && = 1 \\ \Leftrightarrow\ & (st)^{n-m} && = 1 \end{aligned}$$

with $n - m < n$, a contradiction. □

Example 1.8 (Infinite dihedral groups)**.** Let s_1 and s_2 be the reflections of the real line $\mathbb{R}$ in the points 0 and 1, respectively. (Note any point on the line is a hyperplane.)

The composition $s_1 s_2$ is a translation by 2 units to the left; hence $\langle s_1, s_2 \rangle$ is infinite cyclic. The group generated by these reflections has the presentation:

$$W := D_\infty = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1 \rangle$$

and Coxeter matrix $\begin{pmatrix} 1 & \infty \\ \infty & 1 \end{pmatrix}$. The action of W on $\mathbb{R}^1$ induces a tesselation of the line by closed intervals which are in bijection with the elements of W.

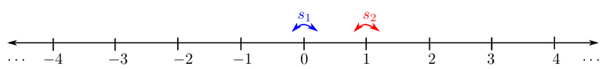


Figure 1: The infinite dihedral group

Example 1.9 (Euclidean triangle group)**.** Let s_1, s_2 and s_3 be the reflections in the plane $\mathbb{R}^2$ through sides of equilateral triangles. The composition $s_1 s_2$ is a flip upwards followed by a flip to the left. The group generated by these reflections has the presentation:

$$W := (3,3,3) = \langle s_1, s_2, s_3 \mid s_i^2 = 1\ \forall i,\ (s_i s_j)^3 = 1\ \forall i \neq j \rangle$$

and Coxeter matrix $\begin{pmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix}$. W induces a tesselation of the plane by equilateral triangles which are in bijection with the elements of W.

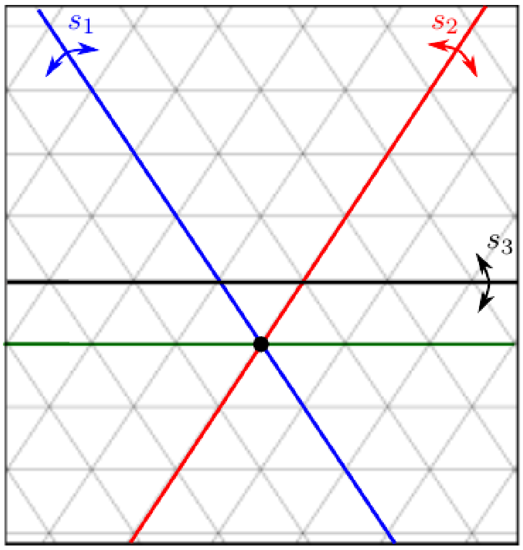


Figure 2: The Euclidean triangle group $(3,3,3)$

1.1 Coxeter diagrams

Before giving further examples of Coxeter groups, we define the *Coxeter graph* which allows to directly read properties of Coxeter groups from a certain graph.

Definition 1.10 ([Davis, §3.5.1])**.** Suppose that $M = (m_{ij})_{i,j\in I}$ is a Coxeter matrix on a set I. We associate to M a graph $\Gamma = \Gamma_M$ called its **Coxeter graph**. The vertex set of Γ is I, representing generators $(s_i)_{i\in I}$.

- Distinct vertices i and j are connected by an edge if and only if $m_{ij} \geq 3$.
- The edge $\{i, j\}$ is labeled by $m_{ij} \geq 4$. (If $m_{ij} = 3$, the edge is left unlabeled.)

The graph Γ *together with the labeling* of its edges is called the **Coxeter diagram** associated to M. The vertices of Γ are often called the **nodes** of the diagram.

Example 1.11.

- The dihedral group D_{2m} of order $2m$ (the isometry group of a regular polygon with $2m$ sides) has Coxeter matrix and diagram:

$$\begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix}, \qquad \bullet \overset{m}{\text{———}} \bullet \text{ if } m \geq 4, \text{ or } \bullet\text{———}\bullet \text{ if } m = 3,$$

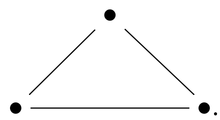
$$\text{or } \bullet \qquad\quad \bullet \text{ if } m = 2.$$

- The infinite dihedral group D_∞ is given by:

$$\begin{pmatrix} 1 & \infty \\ \infty & 1 \end{pmatrix}, \qquad \bullet \overset{\infty}{\longrule{}} \bullet.$$

- The $(3,3,3)$ triangle group is given by:

$$\begin{pmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix},$$



- ([Suter, p.8]) The Coxeter system with diagram:

$$\bullet \longrule{} \bullet \overset{\infty}{\longrule{}} \bullet$$

  is isomorphic to $PGL_2(\mathbb{Z})$, defined as the quotient group

$$GL_2(\mathbb{Z}) / \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \right\}$$

  with generators:

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Remark 1.12 ([Thomas, 1.19.4]). A given Coxeter *group* may have more than one (conjugacy class of) generating set, that is, more than one Coxeter *system*. This is reflected by different Coxeter *diagrams*, as the following example shows.

Example 1.13. Let $W = D_{12}$ be the isometry group of a 12-gon. We consider the following diagrams:

$$\begin{pmatrix} 1 & 6 \\ 6 & 1 \end{pmatrix}, \qquad \bullet_1 \overset{6}{\longrule{}} \bullet_2 \, , \tag{1}$$

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}, \qquad \bullet_1 \overset{3}{\longrule{}} \bullet_2 \qquad \bullet_3 \, . \tag{2}$$

In the first diagram, the element $w = (s_1 s_2)^3$, a rotation by the angle π about the origin, is a *central* involution. (That is, an involution which commutes with all elements in W.)

The group (1) then splits as the direct product of $\langle w \rangle \cong \mathbb{Z}_2$ and a copy of D_6, generated by the reflections s_1 and $s_2 s_1 s_2$. Setting $t_1 = s_1$, $t_2 = s_2 s_1 s_2$ and $t_3 = w$ results in the group (2).

Remark 1.14. By replacing an axis of reflection with an axis of rotation, we get a third presentation of the D_{12}:

$$G = \langle s, t \mid s^6 = 1,\ st = ts^{-1} \rangle.$$

The finite presentation G is *not* a Coxeter system. We can write it as the following semi-direct product: [Loeh, Example 2.3.5]

$$\begin{aligned} D_{12} &\longleftrightarrow \mathbb{Z}_6 \rtimes_\varphi \mathbb{Z}_2 \\ s &\longmapsto ([1], 0) \\ t &\longmapsto (0, [1]), \end{aligned}$$

where $\varphi : \mathbb{Z}/2\mathbb{Z} \to \operatorname{Aut} {}^{\mathbb{Z}}\!/_{2\mathbb{Z}}$ is given by multiplication by -1. Similarly, the dihedral group D_∞ is isomorphic to $\mathbb{Z} \rtimes_\varphi \mathbb{Z}_2$.

Definition 1.15. We call a Coxeter system (W, S) **irreducible** if the Coxeter graph is connected.

Remark 1.16. If W is *reducible* with connected components I and J, then W allows a direct product composition:

$$W_T \times W_{T'}, \quad T = (s_i)_{i \in I}, \quad T' = (s_j)_{j \in J},$$

where the subgroups W_T and $W_{T'}$ in W are generated by T and T', respectively.

We delay the proof of this remark to section 3, after we have established properties on subgroups of Coxeter systems. A simple example is given by $D_\infty \times D_\infty$, which has Coxeter diagram:

$$\bullet \overset{\infty}{\text{———}} \bullet \qquad \bullet \overset{\infty}{\text{———}} \bullet$$

1.2 The length function

In this section, we consider the *length function* of a word in a Coxeter system (W, S). As a method commonly used for establishing properties of Coxeter systems, we demonstrate some of its basic properties.

Definition 1.17. Let (W, S) be a Coxeter system. We define the **length function**

$$\begin{aligned} \ell : W &\longrightarrow \mathbb{Z}_{\geq 0} \\ w &\longmapsto \min\{n \mid \exists s_1, \ldots, s_n \in S \text{ with } w = s_1 \cdots s_n\}. \end{aligned}$$

By definition, $\ell_S(1) = 0$. If $\ell_S(w) = n \geq 1$ and $w = s_1 \cdots s_n$ then the corresponding word $(s_1, \ldots, s_n)$ is variously called a **reduced expression**, a **reduced word** or a **minimal word** for g.

Proposition 1.18 ([Bjorner, Proposition 1.4.2]). *Let (W, S) be a Coxeter system with $u, w \in W$. Let $\varepsilon : W \to \mathbb{Z}/2\mathbb{Z}$, $s \mapsto -1$ be the epimorphism from Lemma 1.6. The length function satisfies the following properties.*

1. $\varepsilon(w) = (-1)^{\ell(w)}$.
2. $\ell(uw) \equiv \ell(u) + \ell(w) \mod 2$.
3. $\ell(w^{-1}) = \ell(w)$.
4. $|\ell(u) - \ell(w)| \leq \ell(uw) \leq \ell(u) + \ell(w)$.
5. $\ell(ws) = \ell(w) \pm 1$, *for all* $s \in S$.
6. $\ell(u^{-1}w)$ *is a metric on* W, *the* ***word metric*** d_S.
7. $d_S(hw, hw') = d_S(w, w')$ *for all* $h, w, w' \in W$.

   That is, the left action of W *on itself is an action by isometries with respect to the word metric* d_S.

Proof.

1. Let $w = s_1 \cdots s_n$ be a reduced expression in W. As ε is a homomorphism, there holds $\varepsilon(w) = \varepsilon(s_1) \cdots \varepsilon(s_n) = (-1)^n$.
2. Let $u = s_1 \cdots s_n$ and $w = t_1 \cdots t_m$ be reduced expressions in W. Then $(-1)^{\ell(uw)} = \varepsilon(uw) = \varepsilon(u)\varepsilon(w) = (-1)^{\ell(u)}(-1)^{\ell(w)} = (-1)^{\ell(u)+\ell(w)}$ or equivalently, $\ell(uw) \equiv \ell(u) + \ell(w) \mod 2$.
3. If $w = s_1 \cdots s_n$ is a reduced expression, then $w^{-1} = (s_1 \cdots s_n)^{-1} = s_n^{-1} \cdots s_1^{-1} = s_n \cdots s_1$ as each s_i is an involution. Therefore $\ell(w^{-1}) \leq n = \ell(w)$. Similarly, if $w^{-1} = t_1 \cdots t_m$ is a reduced expression, then $(w^{-1})^{-1} = w = t_m \cdots t_1$, and $\ell(w) \leq m = \ell(w^{-1})$.
4. Let $u = s_1 \cdots s_n$ and $w = t_1 \cdots t_m$ be reduced expressions in W. As $uw = s_1 \cdots s_n t_1 \cdots t_m$ there holds $\ell(uw) \leq n + m = \ell(u) + \ell(w)$. In particular, $\ell(u) = \ell(((uw)w^{-1}) \leq \ell(uw) + \ell(w^{-1})$, and by 4. $|\ell(u) - \ell(w)| \leq \ell(uw)$.
5. By 4. there holds $\ell(w) - 1 = \ell(w) - \ell(s) \leq \ell(ws) \leq \ell(w) + \ell(s) = \ell(w) + 1$. Assume $\ell(ws) = \ell(w)$. Then $ws = s_1 \cdots s_n s = t_1 \cdots t_n$ with $t_i \in S$. As w was reduced, we have $\varepsilon(ws) = \varepsilon(w)\varepsilon(s) = (-1)^{n+1}$, but $\varepsilon(t_1 \cdots t_n) = (-1)^n$; a contradiction. Thus $\ell(ws) = \ell(w) - 1$ or $\ell(ws) = \ell(w) + 1$.
6. Let $d_S(u, w) := \ell(u^{-1}w)$. If $u \neq w$, then $u^{-1}w \neq 1$ and $\ell(u^{-1}w) > 0$ by definition, so d_S is positive definite. For symmetry, $d_S(w, u) = \ell(w^{-1}u) \leq \ell(w^{-1}) + \ell(u) = \ell(w) + \ell(u)$ and $d_S(u, w) = \ell(u^{-1}w) \leq$

$\ell(u) + \ell(w)$ by 3, thus $d_S(w, u) = d_S(u, w)$. For the triangle inequality, by 4. there holds $d_S(u, w) = \ell(u^{-1}zz^{-1}w) \leq \ell(u^{-1}z) + \ell(z^{-1}w) = d_S(u, z) + d_S(z, w)$ for any $z \in W$.

7. There holds $d_S(hw, hw') = \ell(w^{-1}h^{-1}hw') = \ell(w^{-1}w') = d_S(w, w')$.

□

1.2.1 The Exchange Condition

Two important combinatorial properties of Coxeter systems are given by the *Exchange Condition* and *Deletion Condition.* They will allow us to derive properties of *special subgroups* of (W, S), or groups W_T generated by subsets $T \subset S$. (Section 3)

Theorem 1.19. *Suppose a group W is generated by a set of distinct involutions S. Then the following are equivalent:*

1. *The pair (W, S) is a Coxeter system.*
2. *The pair (W, S) satisfies the **deletion condition**:*

   If $(s_1, \ldots, s_k)$ is a word in S with $\ell(s_1 \cdots s_k) < k$, then there are indicies $i < j$ such that

   $$s_1 \cdots s_k = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k,$$

   where $\hat{s}_i$ means we delete this letter.
3. *Te pair (W, S) satisfies the **exchange condition**:*

   If $(s_1, \ldots, s_k)$ is a reduced expression for $w \in W$, then for any $s \in S$, either $\ell(sw) = k + 1$, or there is an index i such that

   $$w = ss_1 \cdots \hat{s}_i \cdots s_k.$$

There are several ways to prove the above theorem. For a purely combinatorial proof, see [Bjorner, Theorem 1.5.1]. For an approach using *Cayley graphs*, see [Thomas, Theorem 2.14]. A purely geometric approach can be achieved with *van-Kampen diagrams*, where group relations are presented through a certain 2-complex. See [Ol'shanskii, 4. Diagrams over groups] and [Bahls, 1.3.4. The Deletion Condition] for an introduction to this topic.

Remark. The equivalence above holds for groups *generated by involutions*. There exist groups generated by elements of infinite order satisfy the Deletion Condition, or *Artin groups.* [Bahls, Exercise 17]

We can derive the following properties from the Deletion and Exchange condition:

Corollary 1.20 ([Bjorner, Corollary 1.4.8]). *Let (W, S) be a Coxeter system.*

1. *Any expression $w = s_1 \cdots s_k$ contains a reduced expression for w as a subword, obtainable by deleting an even number of letters.*

2. *Suppose $w = s_1 s_2 \cdots s_k = s'_1 s'_2 \cdots s'_k$ are two reduced expressions. Then, the set of letters appearing in the word $s_1 s_2 \cdots s_k$ equals the set of letters appearing in $s'_1 s'_2 \cdots s'_k$.*

3. *S is a minimal generating set for W. That is, no Coxeter generator can be expressed in terms of the others.*

1.3 Finite Coxeter groups

In our later discussion of the *Davis complex* Σ corresponding to a Coxeter group (W, S), we wish to derive, directly from the Coxeter diagram $\Gamma(W, S)$, if Σ has certain geometric properties. (In particular, if Σ has *isolated flats*; see Definition 6.7.) The first step to achieve this is classifying the *finite*, *affine* and *hyperbolic* Coxeter groups.

Proposition 1.21 ([Bjorner, Exercise 1.4]). *Let (W, S) be a finite, irreducible Coxeter system. The Coxeter diagram Γ satisfies the following requirements:*

1. *Γ is a tree.*

2. *Γ has at most one vertex of degree 3 and none of higher degree.*

3. *Γ has at most one marked (i.e., label ≥ 4) edge.*

4. *If Γ has a degree 3 vertex, then all edges are unmarked.*

Example. The triangle group $(3, 3, 3)$ is an *infinite* irreducible Coxeter system, with a circuit as Coxeter diagram.

We can prove the above properties, as well as classify *all* finite Coxeter groups, using the *Cosine matrix* associated to a Coxeter matrix M. [Humphreys, 2.7 Classification of graphs of positive type] We will discuss this together with the *affine* Coxeter groups.

An alternative, purely combinatorial approach to proving Proposition 1.21 are so-called *braid moves*. They were used by J. Tits to find a solution to the *word problem*[1] in Coxeter groups.

[1] *The **conjugation problem** for a finitely generated group G asks for the existence of an algorithm which can determine if two elements in G are conjugates. The **word problem** considers if any given element in G is the neutral element. [Loeh, Definition 7.4.1]*

1.3.1 Braid moves

If $m_{st} < \infty$ represents the order of st in a Coxeter system (W, S), then clearly $sts\cdots = tst\cdots$ for any word of length m_{st}. The terminology "braid move" comes from the defining relation in the presentation

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

of the braid group on three strands. See [Loeh, Exercise 2.E.26] for details on braid groups.

Definition 1.22 ([Thomas, Definition 2.20])**.** Let W be a group generated by a set of involutions S, and let m_{st} be the order of st, $s \neq t$ in W. If $m(s,t)$ is finite, a **braid move** on a word $w \in W$ swaps a subword $(s, t, s, \dots)$ containing m_{st} letters with a subword $(t, s, t, \dots)$ containing m_{st} letters.

Theorem 1.23 ([Davis, Theorem 3.4.2])**.** *(Tits) Suppose a group W is generated by a set of distinct involutions S and the exchange condition holds.*

1. *A word $(s_1, \dots, s_k)$ in S is reduced if and only if it cannot be shortened by a sequence of*
   (a) deleting a subword (s, s), $s \in S$, or
   (b) carrying out a braid move.
2. *Two reduced expressions in S represent the same group element $w \in W$ if and only if they are related by a finite sequence of braid moves.*

Example. Let $(W, \{s_1, s_2, s_3\})$ be the Coxeter system with diagram $\bullet - \bullet - \bullet$. The possible braid moves in W are given by:

$$\begin{aligned}
(s_1, s_3) &\leftrightarrow (s_3, s_1) \\
(s_1, s_2, s_1) &\leftrightarrow (s_2, s_1, s_2) \\
(s_2, s_3, s_2) &\leftrightarrow (s_3, s_2, s_3)
\end{aligned}$$

Then the word $(s_1 s_2 s_3)$ has order 4, as can be seen by carrying out braid moves:

$$\begin{aligned}
s_1 s_2 \mathbf{s_3}\ \mathbf{s_1} s_2 s_3\ s_1 s_2 s_3 &\to \mathbf{s_1 s_2 s_1}\ \mathbf{s_3 s_2 s_3}\ s_1 s_2 s_3 \\
&\to s_2 s_1 \underbrace{s_2\ s_2}_{=1} s_3 s_2\ s_1 s_2 s_3 \\
&\to \cdots \\
&\to \underbrace{s_2 s_2}_{=1} s_3 s_2 s_1 \\
\Leftrightarrow (s_1 s_2 s_3)^4 = s_3 s_2 s_1 s_1 s_2 s_3 = 1
\end{aligned}$$

For illustrative purposes, we use braid moves to show that for a finite, irreducible Coxeter system, the diagram $\Gamma(W, S)$ is a tree. The idea is to find an element of infinite order. The other properties in Proposition 1.21 can be proved similarly.

Proof of Proposition 1.21.1. Let (W, S) be a finite and irreducible Coxeter system. By definition, $\Gamma(W, S)$ is connected. Assume by contradiction that Γ contains a circuit $(i, i+1, \dots, i+n)$, $n \geq 2$. Assume the element $w := s_i s_{i+1} \cdots s_{i+n}$ is of finite order. Then there is some $m \in \mathbb{N}$ such that $w^m = 1$. The possible braid moves in W are given by:

$$(s_j, s_{j+1}, s_j, \cdots) \leftrightarrow (s_{j+1}, s_j, s_{j+1}, \cdots)$$

for subwords with $m_{s_j s_{j+1}}$ letters. By assumption, $s_i s_{i+k} \neq s_{i+k} s_i$ for all $k \in \mathbb{N}$ (otherwise $m(s_i, s_{i+k}) = 2$ and s_i, s_{i+k} are not connected by an edge). Thus we can perform no braid move on the word w^m. In particular, $\ell(w^m) > 1$; a contradiction. ☐

1.4 Affine Coxeter groups

For the sake of brevity, we consider affine Coxeter groups as *(cocompact) Euclidean reflection groups* generated by affine transformations in the Euclidean space $\mathbb{E}^n$.[2] We first recall some results on group actions.

1.4.1 Geometric actions

Coxeter groups are defined as groups with a certain finite presentation. It is known that every finitely presented group acts *geometrically* (that is, properly and cocompactly by isometries) on some simply-connected, geodesic space. The Davis complex is an example of such a space. Reversely, every group with such an action is finitely presented. [Bridson, §8.11]

Definition 1.24 ([Davis, Definition 5.1.5])**.** Suppose G is discrete. A G-action on a Hausdorff space Y is **proper** (or **properly discontinuous**) if the following three conditions hold.

1. Y/G is Hausdorff.
2. For each $y \in Y$, the isotropy subgroup $G_y = \{g \in G \mid gy = y\}$ is finite.
3. Each $y \in Y$ has a G_y-stable neighborhood U_y such that $gU_y \cap U_y = \emptyset$ for all $g \in G - G_y$.

[2] The affine Coxeter groups are (up to the choice of a root system) the *affine Weyl groups*, defined through *coroot lattices*. See [Humphreys, 4. Affine reflection groups] for details on this construction. A nice visual representation is through so-called *Stiefel diagrams*. [Hall, 13.6 The Stiefel Diagram]

Definition 1.25. Let G be a group acting on a topological space X. The action is **cocompact** if the quotient space $G\backslash X$ is compact with respect to the quotient topology.

Definition 1.26. Suppose a group G acts on a topological space X by homeomorphisms. Write Gx for the G-orbit of the point $x \in X$. A **fundamental domain** is a closed, connected subset C of X such that $Gx \cap C \neq \emptyset$ for every $x \in X$, and $Gx \cap C = \{x\}$ for every x in the interior of C. A fundamental domain C is **strict** if $Gx \cap C = \{x\}$ for every $x \in C$, that is, C contains exactly one point from each G-orbit.

Example 1.27 ([Thomas, Example 1.8])**.** The closed interval $[0, 1]$ is a strict fundamental domain for the action of D_∞ on the real line. (Example 1.8) Any interval $[n, n+1]$, where $n \in \mathbb{Z}$, is also a strict fundamental domain for this action.

1.4.2 Geometric reflection groups

Theorem 1.28 ([Thomas, Theorem 1.9])**.** *Let $P = P^n$ be a simple convex polytope in $\mathbb{X}^n$, where $n \geq 2$ and $\mathbb{X}^n = \mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$. Let $\{F_i\}_{i\in I}$ be the collection of codimension-1 faces of P^n, with each face F_i supported by the hyperplane $\mathcal{H}_i$.*

Suppose that for all $i \neq j$, if $F_i \cap F_j \neq \emptyset$ then the dihedral angle between F_i and F_j ist $\frac{\pi}{m_{ij}}$ for some integer $m_{ij} \geq 2$. Put $m_{ii} = 1$ for every $i \in I$, and $m_{ij} = \infty$ if $F_i \cap F_j = \emptyset$.

For each $i \in I$, let s_i be the isometric reflection of $\mathbb{X}^n$ across the hyperplane $\mathcal{H}_i$. Let W be the group generated by the set of reflections $\{s_i\}_{i\in I}$. Then:

1. *The group W has presentation*
$$W \cong \langle s_i \mid (s_i s_j)^{m_{ij}} = 1 \ \forall i, j \in I\rangle.$$
2. *The group W is a discrete subgroup of Isom$(\mathbb{X}^n)$.*
3. *The convex polytope P is a strict fundamental domain for the action of W on $\mathbb{X}^n$, and the action of W induces a tesselation of $\mathbb{X}^n$ by copies of P.*

Example 1.29. For more examples, see [Thomas, pp. 10-15].

- If P is the closed interval $[0, 1]$, then W is the infinite dihedral group D_∞.
- If P is a triangle with vertex angles $\frac{\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{r}$, W is the triangle group (p, q, r).

- There is a convex polytope $P \subset \mathbb{H}^3$ which is a dodecahedron with all dihedral angles $\frac{\pi}{2}$, hence a hyperbolic reflection group W generated by the reflections in the sides of P.

  We can give a presentation of W as follows. The dodecahedron P has 12 sides, thus $S = \{s_1, \ldots, s_{12}\}$. Each face F_i of P has precisely 5 adjacent faces $F_{j_1}, \ldots, F_{j_5}$. (Compare Figure 4.) By Theorem 1.28, it follows that $m_{ij_k} = 2$ for $k \in \{1, \ldots, 5\}$ (the intersection $F_i \cap F_{j_k}$ is an edge, thus non-empty), and $m_{ij} = \infty$ otherwise (F_i and F_j are not adjacent, or $F_i \cap F_j = \emptyset$).

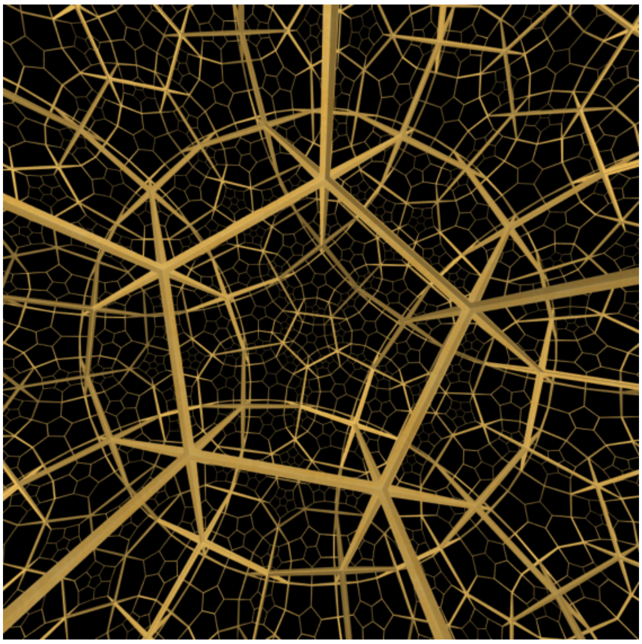


Figure 3: Tiling of $\mathbb{H}^3$ by right-angled dodecahedra.

Definition 1.30. A group W is a *geometric reflection group* if W is either a finite dihedral group, an infinite dihedral group or is as in the statement of Theorem 1.28. A geometric reflection group W acting on $\mathbb{X}^n$ is **spherical**, **Euclidean** or **hyperbolic** as $\mathbb{X}^n$ is $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ respectively. A group W is an **affine Coxeter group** if W is an Euclidean ($\mathbb{X}^n = \mathbb{E}^n$) geometric reflection group.

1.4.3 Cosine matrix

We turn to the classification of affine Coxeter groups (Euclidean reflection groups) and hyperbolic reflection groups.

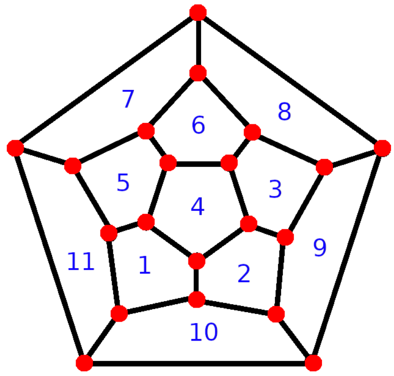


Figure 4: Example *Schlegel diagram* for P. Each face has 5 adjacent faces.

Definition 1.31. Suppose $M = (m_{ij})$ is a Coxeter matrix on a set I. The **cosine matrix** associated to M is the $I \times I$ matrix (c_{ij}) defined by

$$c_{ij} = -\cos\frac{\pi}{m_{ij}}$$

When $m_{ij} = \infty$ we interpret $\frac{\pi}{\infty}$ to be 0, and $-\cos\frac{\pi}{\infty} = -\cos(0) = -1$. Note that all diagonal entries of (c_{ij}) are $-\cos\frac{\pi}{1} = 1$.

Definition 1.32. Let $A = (a_{ij})$ be a square $n \times n$ matrix. The k-th **principal submatrix** of A is the matrix obtained by deleting the k-th row and k-th column of A.

Definition 1.33. Let $A = (a_{ij})$ be a square $n \times n$ matrix. A is **reducible** (or **decomposable**) if there is a nontrivial partition of the index set as $\{1, \ldots, n\} = I \cup J$, so that $a_{ij} = a_{ji} = 0$ whenever $i \in I$, $j \in J$. Otherwise, it is **irreducible** (or **indecomposable**).

Theorem 1.34 ([Davis, Theorem 6.8.12])**.** *Let $M = (m_{ij})$ be a Coxeter matrix over I, W the associated Coxeter group, and $C = (c_{ij})$ the associated cosine matrix. Suppose that no m_{ij} is ∞. Then:*

1. *W can be represented as a spherical reflection group generated by the reflections across the faces of a spherical simplex if and only if C is positive definite.*

2. *Suppose, in addition, that M is irreducible. Then W can be represented as a Euclidean reflection group generated by the reflections across the faces of a Euclidean simplex if and only if C is positive semidefinite of corank 1.*

3. *W can be represented as a hyperbolic reflection group generated by the reflections across the faces of a hyperbolic simplex if and only if C is*

nondegenerate of type $(n, 1)$ and each principal submatrix is positive definite.

We can restate the classification of finite Coxeter groups in terms of the first property.

Theorem 1.35 ([Davis, 6.12.9])**.** *Suppose $M = (m_{ij})$ is a Coxeter matrix on a set I, that (c_{ij}) is its associated cosine matrix, and that (W, S) is its associated Coxeter system. Then the following statements are equivalent:*

1. *W is a reflection group on $\mathbb{S}^n$, $n = |I| - 1$, so that the elements of S are represented as the reflections across the codimension-one faces of a spherical simplex σ.*
2. *(c_{ij}) is positive definite.*
3. *W is finite.*

1.5 Hyperbolic Coxeter groups

Let W be a hyperbolic reflection group where the convex polytope P is a simplex. Such groups exist only in ranks 3 to 10, and there are only finitely many in each of ranks 4 to 10. If the action of W on $\mathbb{H}^n$ is cocompact (W is a *compact hyperbolic group*), such groups exist only in ranks 3, 4 and 5. [Humphreys, 6.9 List of hyperbolic Coxeter groups]

The difference between compact and non-compact hyperbolic groups can be characterized through the Coxeter diagram: if for a Coxeter system (W, S) we remove a vertex (representing a generator $s \in S$) in the diagram $\Gamma(W, S)$, then the resulting diagram either represents a finite (in the compact case) or an affine Coxeter group (in the non-compact case). [Humphreys, 6.8 Hyperbolic Coxeter groups]This leads us to the following definition:

Definition 1.36 ([Caprace])**.** Let (W, S) be a Coxeter system (with S finite). We say that J is **minimal hyperbolic** if it is non-spherical and non-affine, but every proper subset is spherical or irreducible affine.

2 The Tits representation

The Tits representation is a commonly used tool in the theory of Coxeter groups. It allows us to easily verify the order of generators in Coxeter groups, derive properties of special subgroups, and helps define a metric on the *Davis complex*. It may also be used as an aid in the classification of finite Coxeter groups.

Theorem 2.1. *Let (W, S) be a Coxeter system. Let $V := \bigoplus_{s\in S} \mathbb{R}e_s$ be an $|S|$-dimensional real vector space with canonical basis $\{e_s\}_{s\in S}$. Define a symmetric bilinear form B on V by:*

$$B(e_s, e_t) = \begin{cases} -\cos\frac{\pi}{m_{st}} & m_{st} \neq \infty \\ -1 & m_{st} = \infty \end{cases}$$

*There exists a faithful action (the **Tits representation** or **canonical representation**)*

$$\sigma : W \to GL(V)$$
$$s \mapsto (\sigma_s : \lambda \mapsto \lambda - 2B(e_s, \lambda)e_s)$$

where $s \in S$. The map σ_s represents the reflection across e_s to the hyperplane

$$H_s := \{\lambda \in V \mid B(e_s, \lambda) = 0\}$$

and has the following properties:

1. *σ_s is linear with fixed set H_s.*
2. *σ_s preserves the bilinear form B, that is $B(\sigma_s(\lambda), \sigma_s(\mu)) = B(\lambda, \mu)$ for all $\lambda, \mu \in V$.*
3. *$\sigma_s(e_s) = -e_s$.*
4. *$\sigma_s^2 = \mathrm{id}$ for each $s \in S$.*
5. *$\sigma_s\sigma_t$ has order m_{st} for all distinct $s, t \in S$.*

Proof. (Sketch) Properties 1 to 4 are clear. To determine the order of $\sigma_s\sigma_t$, we distinguish the cases $m_{st} < \infty$ and $m_{st} = \infty$ (by definition of B). Let $V_{st} := \mathbb{R}e_s \oplus \mathbb{R}e_t$. The bilinear form B is symmetric, and B is positive definite on V_{st} if and only if $m_{st} < \infty$. In this case, $(V_{st}, B|_{V_{st}})$ is an Euclidean plane, and the subgroup of $GL(V)$ generated by σ_s and σ_t is a dihedral group of order $2m_{st}$. [Elements, V.4.2] When $m_{st} = \infty$, B is positive semidefinite on V_{st}, but $\sigma_s\sigma_t(e_s) = e_s + 2(e_s + e_t)$; by induction, $\sigma_s\sigma_t$ has infinite order on V_{st}.

With $\sigma_s^2 = \mathrm{id}$ and $(\sigma_s\sigma_t)^{m_{st}} = \mathrm{id}$, we extend $S \to GL(V)$, $s \mapsto \sigma_s$ to a homomorphism $\sigma : W \to GL(V)$ by setting $\sigma(w) := \sigma_{s_1} \cdots \sigma_{s_n}$ for $w = s_1 \cdots s_n \in W$.

It remains to show that the action σ is faithful. We proceed by considering the dual representation $\sigma^* : W \to GL(V^*)$, given by

$$(\sigma^*(w)(\varphi))(v) = \varphi(\sigma(w^{-1})(v)),$$

with $\varphi \in V^*$, $w \in W$ and $v \in V$. If the dual σ^* is faithful, then σ is also faithful. The key to proving this is considering **chambers**

$$C := \{\varphi \in V^* \mid \varphi(e_i) \geq 0 \ \forall i \in I\},$$

and their interiors $\mathring{C}$. A theorem by Tits then says that if $\sigma^*(w)\mathring{C} \cap \mathring{C}$ is nonempty, then $w = 1$. The claim then follows: if $\sigma^*(w) = 1$, then $\sigma^*(w)(C) = C$, or $w = 1$. The proof of the theorem uses the *length* ℓ of $w \in W$ (relative to the generating set S). See [Elements, V.4.4] for a full description. For a combinatorial view on the dual representation σ^*, see [Bjorner, 4.3 The numbers game]. $\square$

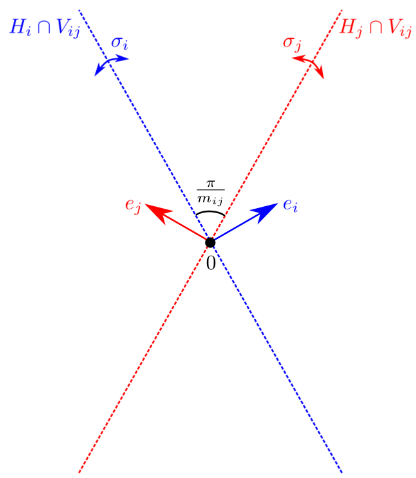


Figure 5: The case $m_{st} < \infty$

An alternative way in proving that the Tits representation is faithful is so-called *root systems*. [Suter, Corollary 4.7]

We are now able to prove the remaining properties in Proposition 1.5.

Proof of 1.5. We have already shown that each generator $s \in S$ is an involution. Denote by w_s the image of s in W. The composition $s \mapsto w_s \mapsto \sigma_s$ is injective by Theorem 2.1.5 (if $s \neq t$, then $\sigma_s\sigma_t$ has order ≥ 2), thus $s \mapsto w_s$ is also injective. This shows that each $s \in S$ is distinct in W.

It remains to show that each st has order m_{st}. An element w_{st} has *at most* order m_{st}, and σ_{st} has *precisely* order m_{st}. It follows that w_{st} has precisely order m_{st}, which shows the claim. $\square$

Remark 2.2. The matrix for the bilinear form $B(e_s, e_t)_{s,t\in S}$ is precisely the Coxeter matrix C defined in Section 1.4.3. In particular, a Coxeter system (W, S) of finite rank is finite, if and only if the canonical bilinear form is positive-definite.

2.1 Coxeter polytopes

The Tits representation gives us a first geometric realisation of (finite) Coxeter groups, the *Coxeter polytopes*. Later on, we will paste together these polytopes to get a piecewise Euclidean geometric realisation of an *arbitrary* Coxeter group, called the Davis complex.

Let (W, S) be a finite Coxeter system with $|S| = n$. By Remark 2.2, we can then identify V^* with Euclidean space $\mathbb{E}^n$, and the chamber C is a closed Euclidean simplicial sector cut out by hyperplanes. Choose a point x in the interior of the chamber C. We call such an x a **generic point** (x is determined by specifying its distance to each of the bounding hyperplanes, i.e. by specifying an element of $(0, \infty)^n$.

Notation. From here on, write $w\mathring{C}$ for $\sigma^*(w)\mathring{C}$, $w \in W$.

Definition 2.3 ([Davis, Definition 7.3.1])**.** Let (W, S) be a finite Coxeter system. A **Coxeter polytope** (or **Coxeter cell**) associated to W is the convex polytope C_W defined as the convex hull of Wx (a generic W-orbit).

Lemma 2.4. *Let C be chamber associated to a finite Coxeter system (W, S). Then W acts simply transitively on the set $W\mathring{C} = \{w\mathring{C} \mid w \in W\}$. In particular, for a generic point $x \in \mathring{C}$, $|Wx| = |W|$.*

Proof. The action of W is transitive by construction. It remains to show that it is free. Assume there are $w \neq w'$ in W such that $w\mathring{C} = w'\mathring{C}$. Then $ww'^{-1}\mathring{C} = w'w'^{-1}\mathring{C} = \mathring{C}$, in particular, $ww'^{-1}\mathring{C} \cap \mathring{C} \neq \emptyset$. By Tits, $ww'^{-1} = 1$; a contradiction. $\square$

Example 2.5.

1. If $W = \mathbb{Z}_2$, then C_W is the interval $[-x, x]$.
2. If $W = D_m$, then C is a $2m$-gon. (It is *regular* if the generic point x is equidistant from the two rays which bound the fundamental sector containing x.)
3. If (W, S) is reducible and decomposes as $W = W_1 \times W_2$, then C_W decomposes as $C_W = C_{W_1} \times C_{W_2}$. In particular, if $W = (\mathbb{Z}_2)^n$, then C_W is a product of intervals. (If x is equidistant from the bounding hyperplanes, then C_W is a regular n-cube.)

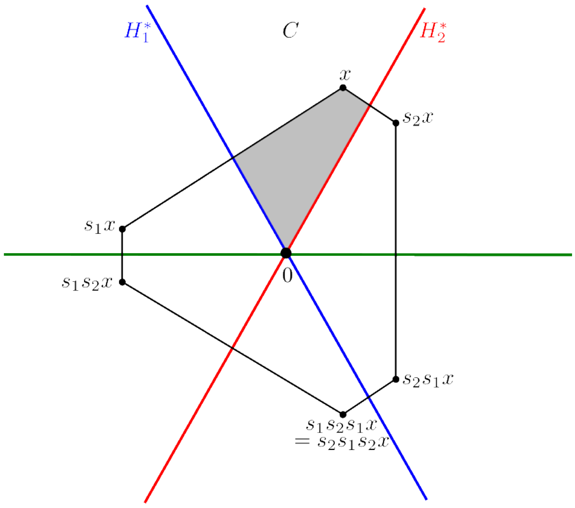


Figure 6: A Coxeter polytope for $W \cong D_6$

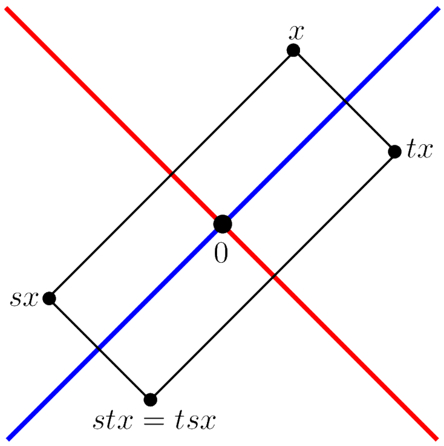


Figure 7: A Coxeter polytope for $W \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

3 Special subgroups

We now turn to subgroups of Coxeter groups and state some of their important properties.

Definition 3.1. For $T \subseteq S$, let W_T be the subgroup of W generated by the set T. We call subgroups of Coxeter systems (W, S) of this form **special subgroups**.[3] If the subgroup W_T is finite, we call it **spherical**.[4]

We show the following properties of special subgroups.

Proposition 3.2 ([Bjorner, Proposition 2.4.1])**.** *Let (W, S) be a Coxeter system, and W_T resp. $W_{T'}$ the subgroups generated by $T \subseteq S$ resp. $T' \subseteq S$. There holds:*

1. *(W_T, T) is a Coxeter system.*
2. *$\ell_T(w) = \ell(w)$ for all $w \in W_T$, where $\ell_T(w)$ denotes the length function with respect to W_T.*
3. *$W_T \cap W_{T'} = W_{T \cap T'}$.*
4. *$\langle W_T \cup W_{T'} \rangle = W_{T \cup T'}$.*
5. *If $W_T = W_{T'}$, then $T = T'$.*

Proof.

1. [Humphreys, 5.5 Parabolic subgroups] Let M be the Coxeter matrix associated to the Coxeter system (W, S) with $S = \{s_j\}_{j \in J}$. Let $T = \{t_i\}_{i \in I}$, $I \subseteq J$ be a subset of S. We define $(\widetilde{W_T}, \tilde{T})$ as the Coxeter system associated to the (restricted) Coxeter matrix $M|_{J \times J}$, with $\tilde{T} = \{\tilde{t}\}_{j \in J}$ a copy of T.

   If (W_T, T) is a Coxeter system, then the epimorphism $\widetilde{W_T} \twoheadrightarrow W_T$ (sending a word $\tilde{t} = \tilde{t}_1 \cdots \tilde{t}_n$ in $\widetilde{W_T}$ to an element $t = t_1 \cdots t_n$ in W_T) must be an isomorphism.

   Let σ_T denote the canonical representation for $\widetilde{W_T}$ with $V_T \subseteq V$ and σ the canonical representation for W. We have the following diagram:

$$\begin{tikzcd}
\widetilde{W_T} \arrow[r, hook, "\sigma_T"] \arrow[dr] & GL(V_T) \arrow[r, hook] & GL(V) \\
 & W_T \arrow[u, "\sigma|_{W_T}"] \arrow[r, hook] & W \arrow[u, hook, "\sigma"]
\end{tikzcd}$$

[3]These groups are often called "(standard) parabolic", after Bourbaki and relations to parabolic subgroups in Lie theory resp. algebraic groups. We call them "special", as the strong algebraic properties in Proposition 3.2 do not hold for (finitely presented) groups in general.

[4]The reason for the term "spherical" is that if (W, S) is an irreducible Coxeter system with W finite, then W acts naturally on the sphere. [Davis, Theorem 6.12.9]

The right diagram commutes, thus $\sigma|_{W_T}$ is a monomorphism. The left diagram also commutes, thus it follows that $\widetilde{W_T} \to W_T$ is an isomorphism, and that (W_T, T) is a Coxeter system.

2. Let $w \in W_T$. Let $w = s_1 \cdots s_q$ with $\ell_T(w) = q$ and $s_1, \ldots, s_q \in T$. By Theorem 1.19, if $\ell(w) < q$ there are indices $i < j$ such that $s_1 \cdots s_q = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$. Since all s_i were already in T and $\ell_T(w)$ is minimal, $\ell(w) = q$ must hold.

3. The inclusion $W_{T \cap T'} \subseteq W_T \cap W_{T'}$ is clear. Conversely, let $w \in W_T \cap W_{T'}$. Then w has reduced expressions $w = s_1 \cdots s_n$, $s_i \in T$ and $w = t_1 \cdots t_m$, $t_j \in T'$ in W_T and $W_{T'}$, respectively By 2, $\ell_T(w) = \ell(w) = \ell_{T'}(w)$, thus $n = m$. By Corollary 1.20.2, the set of letters in (t_j) matches the set of letters in (s_i), and $w = s_1 \cdots s_n$, $s_i \in T \cap T'$ is a reduced expression in $W_{T \cap T'}$.

4. As every element in $\langle T \cup T' \rangle$ is a finite product of elements in $T \cup T'$, the claim follows.

5. Let $T \neq T'$ be subsets of S, with $s \in T$ and $s \notin T'$. If $s \in W_{T'}$, then by 2. there holds $\ell(s) = 1 = \ell_{T'}(s)$, such that $s \in T'$. Therefore $s \notin W_J$.

$\square$

Corollary 3.3. *Let (W, S) be a Coxeter system. The assignment of W_I to I defines a bijective, inclusion-preserving map between the collection of subsets of S and the collection of subgroups W_I of W. In particular, the partially ordered set of subsets of S is isomorphic to the partially ordered set of special subgroups of W.*

Corollary 3.4 ([Davis, Theorem 4.1.6.(iii)])**.** *Let (W, S) be a Coxeter system. Let T, T' be subsets of S and w, w' elements of W. Then $wW_T \subset w'W_{T'}$ (resp. $wW_T = w'W_{T'}$) if and only if $w^{-1}w' \in W_{T'}$ and $T \subset T'$ (resp. $T = T'$).*

Parabolic closure By Proposition 3.2, the subgroup generated by the intersection of subsets $T \cap T' \subset S$ is equal to the subgroup $W_T \cap W_{T'}$. That means we can define the *special closure* (or parabolic closure) of an arbitrary subset $R \subset W$:

Definition 3.5. Let (W, S) be a Coxeter system, and $R \subset W$ a subset. The **special closure** of R is defined as the smallest special subgroup of W containing R:

$$\mathrm{Pc}(R) \coloneqq \bigcap_{T \subset S,\, R \subset W_T} W_T.$$

3.1 Diagrams for special subgroups

We now consider Coxeter diagrams for special subgroups. The Coxeter diagram for (W_T, T) is obtained by removing all nodes in $S\backslash T$ and their incident edges from the diagram for (W, S). Consider paths of shortest length (*geodesic paths*) joining nodes in T to nodes in $S\backslash T$. The number of edges of such a path defines the *distance* between these nodes (or ∞ if there is no such path.)

We can make this statement made precise through the set $J^{\perp}$. Before defining this set, we prove Remark 1.16.

Remark 3.6. If there is no edge between i and j in the Coxeter diagram, then $m_{ij} = 2$ (or $s_i s_j = s_j s_i$) by definition. This implies that if i and j are in different connected components of the Coxeter diagram, they commute.

Remark (1.16). If W is *reducible* with connected components I and J, then W allows a direct product composition:

$$W_T \times W_{T'}, \quad T = (s_i)_{i \in I}, \quad T' = (s_j)_{j \in J},$$

where the subgroups W_T and $W_{T'}$ in W are generated by T and T', respectively.

Proof of Remark 1.16. By assumption, T and T' consist of elements in the connected components I and J, respectively. Thus, $T \cap T' = \emptyset$ and $T \cup T' = S$. By Proposition 3.2 there then holds:

$$\begin{aligned} W_T \cap W_{T'} &= W_{T \cap T'} = W_{\emptyset} = \{1\}, \\ \langle W_T \cup W_{T'} \rangle &= W_{T \cup T'} = W. \end{aligned}$$

Since elements of T and T' commute by Remark 3.6, we have $W \cong W_T \times W_{T'}$. □

Definition 3.7. Let (W, S) be a Coxeter group and $T \subset S$ a subset. We define:

$$J^{\perp} := (S\backslash T) \cap \mathcal{Z}_W(W_T) = \{s \in S\backslash T \mid sw = ws, \ \forall w \in W_T\},$$

where $\mathcal{Z}_W(W_T)$ denotes the centralizer of W_T in W.

Lemma 3.8. *Generators $s \in S$ commute with all group elements $w \in W$ if and only if they commute with all generators $t \in S$. It follows that we may rewrite $J^{\perp}$ as:*

$$J^{\perp} = \{s \in S\backslash T \mid st = ts, \ \forall t \in T\}$$

Proof. For the first statement, proceed by induction of the length of w. By Remark 3.6, these are exactly the vertices $s \in S\backslash T$ in the Coxeter diagram with distance > 1 to vertices in T. □

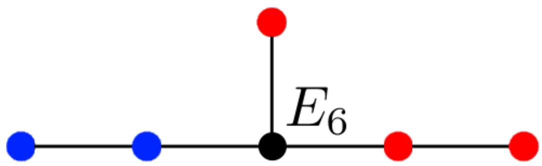


Figure 8: The Coxeter group E_6 (vertices of T in blue, vertices of $J^{\perp}$ in red)

4 Cayley graphs

In Section 5.2, we will define the 1-skeleton as the *Cayley graph* of a the system (W, S). Here, we explain what Cayley graphs are, and how they can be metricized.

Note. For an overview on (combinatorial) graphs, see [Groupes, IV. Annexe]. For details on Cayley graphs, see [Loeh, 3. Cayley graphs] or [Thomas, 2.2 Cayley graphs of Coxeter systems].

Definition 4.1. Let G be a group and let $S \subset G$ be a generating set of G. Then the **Cayley graph** of G with respect to the generating set S is the graph $\mathrm{Cay}(G, S)$ whose set of vertices is G, and whose set of edges is

$$\{\{g, gs\} \mid g \in G,\, s \in (S \cup S^{-1}) \backslash \{e\}\}.$$

If $s \in S$ is an involution, put a single undirected edge between $g \in G$ and $gs = gs^{-1}$.

Remark 4.2 (Action of G on $\mathrm{Cay}(G, S)$).

1. The word metric d_S on G (see Proposition 1.18) extends to the *path metric* on $\mathrm{Cay}(G, S)$, that is, the metric in which each edge of $\mathrm{Cay}(G, S)$ is a unit interval, and the distance between any two points in the graph is given by the length of a shortest path between them.

2. The group G acts by graph isomorphisms on the Cayley graph $\mathrm{Cay}(G, S)$ via left translation:

$$\begin{aligned} G &\longrightarrow \mathrm{Aut}(\mathrm{Cay}(G, S)) \\ g &\longmapsto (h \mapsto g \cdot h); \end{aligned}$$

   This map is well-defined and a group homomorphism. It is an action by isometries with respect to the path metric.

Example 4.3 (Examples of Cayley graphs)**.**

1. The Cayley graph of $D_\infty = \langle s_1, s_2 \mid s_i^2 = 1 \rangle$ is the infinite regular tree of degree 2. (That is, an infinite two-sided path.) The same holds for the Cayley graph of $(\mathbb{Z}, \{1\})$.

2. The Cayley graph of the additive group $\mathbb{Z} \times \mathbb{Z}$ with respect to the generating set $\{(1, 0), (0, 1)\}$ looks like the integer lattice in $\mathbb{R}^2$.

3. The Cayley graph of the cyclic group $\mathbb{Z}/6\mathbb{Z}$ looks like a cycle graph.

4. The Cayley graph of the free group in two generators is the infinite regular tree of degree 4.

   Similarly, the Cayley graph of $D_\infty \times D_\infty \cong (\mathbb{Z}_2 \rtimes_\varepsilon \mathbb{Z})^2$ is dual to the induced tesselation of $\mathbb{R}^2$ by squares.

5. The Cayley graph of the finite dihedral group D_6 (with respect to the generating set $S = \{s_1, s_2\}$) is a hexagon.

6. The Cayley graph of the $(3, 3, 3)$ triangle group (with respect to the set of reflections in the sides of an equilateral triangle in $\mathbb{R}^2$) is dual to the induced tesselation of $\mathbb{R}^2$.

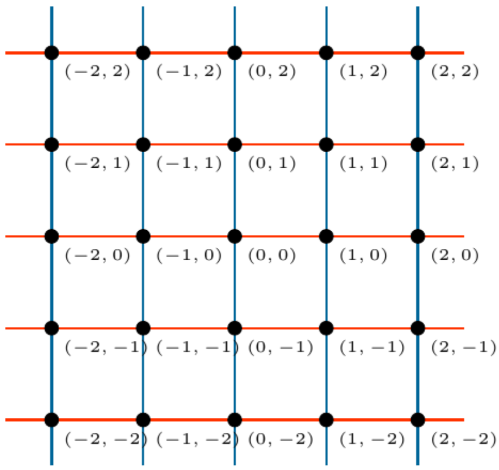


Figure 9: The Cayley graph $\mathrm{Cay}(\mathbb{Z}^2, \{(1, 0), (0, 1)\})$

Remark 4.4 (Properties of Cayley graphs).

- Cayley graphs are connected. Indeed, every vertex g can be reached by the vertex of the neutral element by a path corresponding to a word of minimal length.

- If Cayley graphs are isomorphic as graphs, their corresponding groups are not isomorphic in general. [Loeh, Outlook 3.2.4] (In Example 4.3, both $\mathrm{Cay}(D_\infty, \{s_1, s_2\})$ and $\mathrm{Cay}(\mathbb{Z}, \{1\})$ are a line, but $\mathbb{Z}$ and $D_\infty \cong \mathbb{Z}_2 \rtimes_\varepsilon \mathbb{Z}$ are not isomorphic.)

- The Cayley graph $\mathrm{Cay}(F_S, S)$ of a free group F_S is a tree. (Since each element of F_S can be written uniquely as a reduced word in $S \cup S^{-1}$, there is a unique edge path connecting any given element to 1. Hence, $\mathrm{Cay}(F_S, S)$ contains no circuits.)

  The converse is not true in general. (In Example 4.3, $\mathrm{Cay}(\mathbb{Z}, \{1\})$ is a tree, but $\mathbb{Z}$ is not a free group.)

By definition, there holds:

- Every vertex has the same number $|(S \cup S^{-1}) \backslash \{e\}|$ of neighbours.

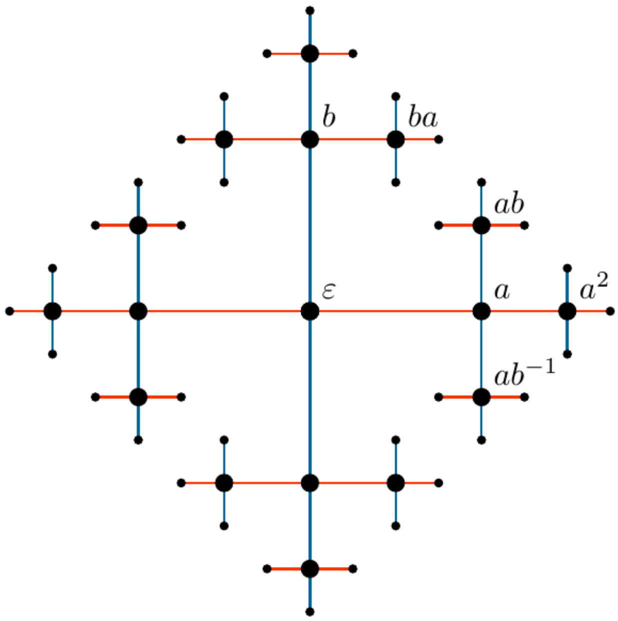


Figure 10: The Cayley graph $\mathrm{Cay}(\langle s_1, s_2 \mid \rangle, \{s_1, s_2\})$

- Cayley graphs are locally finite (that is, every vertex has only finitely many neighbours) if and only if the generating set S is finite.

Definition 4.5. A graph is **simple** if the end points of each edge are distinct vertices (that is, the graph has no loops), and there is at most one edge between any pair of vertices (that is, the graph has no multiple edges).

Lemma 4.6. *Let (W, S) be a Coxeter system. Then $\mathrm{Cay}(W, S)$ is a connected simple graph.*

Proof. It remains to show that $\mathrm{Cay}(W, S)$ is simple. By Proposition 1.5.1, S consists of involutions, hence $1 \notin S$. It follows that $\mathrm{Cay}(W, S)$ has no loops.

By our convention, the edges of $\mathrm{Cay}(W, S)$ are undirected edges of the form $\{w, ws\}$ for $w \in W$ and $s \in S$. By Proposition 1.5.2, the elements of S are pairwise distinct group elements in W, so there is at most one edge between any two vertices of $\mathrm{Cay}(W, S)$. $\square$

4.1 Products in the Cayley graph

Recall that the direct product group $\prod_{i\in I} G_i$ of $(G_i)_{i\in I}$ is the group whose underlying set is the cartesian product $\prod_{i\in I} G_i$, and whose composition is given by pointwise composition $((g_i)_{i\in I}, (h_i)_{i\in I}) \mapsto (g_i \cdot h_i)_{i\in I}$.

The aim of this section is to show that the Cayley graph of a direct product of groups $\prod_{i\in I}(G_i, S_i)$ is the *Cartesian product* of the Cayley graphs

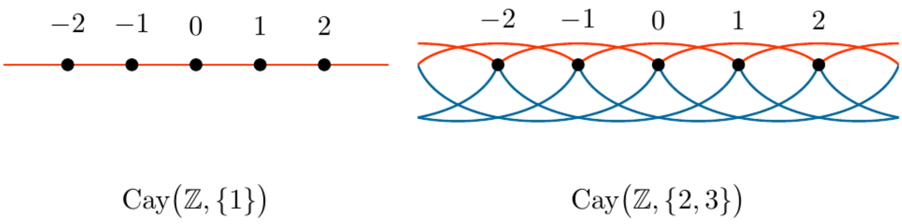


Figure 11: Cayley graphs of the additive group $\mathbb{Z}$

$\mathrm{Cay}(G_i, S_i)_{i\in I}$. We follow [Imrich, 1.4 The Cartesian product] for basic terms and definitions.

Definition 4.7. The **Cartesian product** $\Gamma\Box\Gamma'$ of two graphs Γ and Γ' is defined on the Cartesian product $V(\Gamma) \times V(\Gamma')$ of the vertex set of the factors. The set of edges $E(\Gamma\Box\Gamma')$ is given by:

$$\begin{aligned} E(\Gamma\Box\Gamma') = \{\{(u, v), (x, y)\} \mid u = x, \{v, y\} \in E(\Gamma'),\ \text{or,} \\ \{u, x\} \in E(\Gamma),\ v = y\}. \end{aligned}$$

Since the Cartesian product of graphs is associative (that is, $\Gamma_1\Box(\Gamma_2\Box\Gamma_3) \cong (\Gamma_1\Box\Gamma_2)\Box\Gamma_3$) [Imrich, Proposition 1.36], it suffices to consider products of two graphs.

Lemma 4.8. *Let (W, S) and (W', S') be Coxeter systems with corresponding Cayley graphs $\mathrm{Cay}(W, S)$ and $\mathrm{Cay}(W', S')$.*

Then the Cartesian product $\mathrm{Cay}(W, S)\Box\,\mathrm{Cay}(W', S')$ is given by the Cayley graph of the product $(W \times W', S \sqcup S')$, where $S \sqcup S' \cong \{1\} \times S' \cup S \times \{1\}$.

Proof. First note that $S \sqcup S'$ is a generating set for the group $W \times W$', for W and W' are generated by S and S' respectively, and the composition in $W \times W'$ is pointwise.

By definition, the vertex set of $\mathrm{Cay}(W, S)\Box\,\mathrm{Cay}(W', S')$ is given by:

$$\begin{aligned} V(\mathrm{Cay}(W, S)) \times V(\mathrm{Cay}(W', S')) &= W \times W' \\ &= V(\mathrm{Cay}(W \times W', S \sqcup S'). \end{aligned}$$

The edge set of $\mathrm{Cay}(W, S)\Box\,\mathrm{Cay}(W', S')$ is given by:

$$\begin{aligned} &\{(w, v), (w \cdot s, v) = (w \cdot s, v \cdot 1)\}, && w, v \in W,\ s \in S, \\ &\{(w', v'), (w', v' \cdot s') = (w' \cdot 1, v' \cdot s')\}, && w', v' \in W',\ s' \in S'. \end{aligned}$$

These are precisely the edges of $\mathrm{Cay}(W \times W', \{1\} \times S' \cup S \times \{1\})$.

□

Example 4.9. Let C_6 denote the cyclic graph in 6 vertices, K_2 the complete graph on two vertices (an edge), $K_{1,4}$ the complete bipartite graph on 1 and 4 vertices, and P_3 the path on 3 vertices. Then the products $C_6 \square K_2$ and $K_{1,4} \square P_3$ are given as in Figure 12.

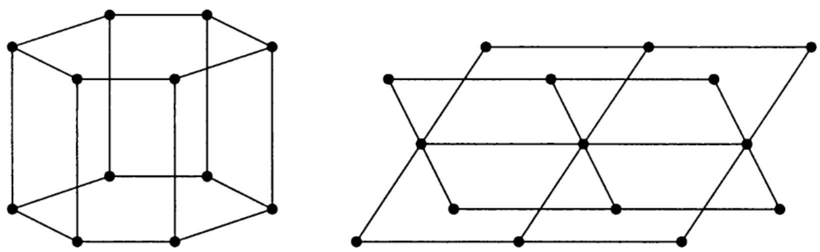


Figure 12: The products $C_6 \square K_2$ and $K_{1,4} \square P_3$

5 The Davis complex

The Davis complex gives an important class of examples for CAT(0) spaces. Each Coxeter system (W, S) has a corresponding Davis complex Σ on which it acts. There are three (up to homeomorphism) equivalent definitions of the Davis complex, which may be used to demonstrate different properties; here we will focus on the definition as *CW complex*.

Let (W, S) be a Coxeter system. The **Davis complex** may be defined as follows:

- As a *basic construction*, a certain quotient space.
- As *geometric realisation* of a partially ordered set.
- As a CW-complex.

The last definition is efficient, in the sense that that there are no "topologically unimportant" cells. We introduce the Davis complex as a partially ordered set and as a CW-complex. For details on the *basic construction*, see [Davis, 5. The Basic Construction].

We show that Σ is a complete CAT(0) space (or *Hadamard space)* using the following properties:

1. Σ is connected.
2. Σ is *simply* connected.
3. Σ is a complete geodesic space. For this we will use the Tits representation of a Coxeter group to define an appropriate metric.
4. Σ is *locally* CAT(0), i.e. of curvature 0.

As Σ is a complete CAT(0) space, we have in particular: (see [Davis, §12.3.4])

1. Σ is contractible.
2. The word and conjugation problems are solvable for W. (Recall that the *conjugation problem* for a finitely generated group G asks for the existence of an algorithm which can determine if two elements in G are conjugates. The *word problem* considers if any given element in G is the neutral element; see Problem 1.1.)

5.1 Geometric realization of a poset

In this section, we introduce the notion of a *geometric realization of a partially ordered set*, a special case of the (standard) geometric realization of an *abstract simplicial complex.* This will give us the first definition of the

Davis complex Σ for a Coxeter system (W, S). For details on simplicial complexes and partially ordered sets, see [Davis, Appendix A] or [Abramenko, Appendix A.1]. For the closely related notion of the *nerve*, see [Davis, 7.1 The Nerve of a Coxeter System].

Definition 5.1. Given a partially ordered set (or poset) $\mathcal{P}$ and en element $p \in \mathcal{P}$, put

$$\mathcal{P}_{\leq p} := \{x \in \mathcal{P} \mid x \leq p\}.$$

Define $\mathcal{P}_{\geq p}$, $\mathcal{P}_{<p}$ and $\mathcal{P}_{>p}$ similarly.

Example 5.2.

- Given a convex polytope P, let $\tilde{\mathcal{F}(P)}$ denote its set of faces (including the empty face), partially ordered by inclusion. Let $\mathcal{F}(P) = \tilde{\mathcal{F}}(P)_{>\emptyset}$ denote the poset of nonempty faces. If Δ^n is an n-simplex, then $\tilde{\mathcal{F}}(\Delta^n) \cong \mathcal{P}(I_{n+1})$, where $\mathcal{P}(I_{n+1})$ denotes the power set of $I_{n+1} = \{1, \ldots, n+1\}$.
- Let (W, S) be a Coxeter system. Recall that a subset T of S is *spherical* if W_T is a finite subgroup of W. Let $\mathcal{S}(W, S)$ (or $\mathcal{S}$) denote the poset of all spherical subsets of S, ordered by inclusion.

Definition 5.3. An **abstract simplicial complex** consists of a set V, possibly infinite, called the **vertex set**, and a collection X of finite subsets of V, such that

1. $\{v\} \in X$ for all $v \in V$; and
2. if $\Delta \in X$ and $\Delta' \subseteq \Delta$, then $\Delta' \in X$.

An element of X is called a **simplex**. If Δ is a simplex and $\Delta' \subsetneq \Delta$, then Δ' is a **face** of Δ. The **dimension** of a simplex Δ is $\dim \Delta = \mathrm{Card}(\Delta) - 1$, and a **$k$-simplex** is a simplex of dimension k. A 0-simplex is sometimes called a **vertex** and a 1-simplex is sometimes called an **edge**.

Example 5.4 ([Davis, Example 7.1.5])**.** Let (W, S) be a Coxeter system and $\mathcal{S}(W, S)$ the poset of spherical subsets. Suppose (W, S) decomposes as

$$(W, S) = (W_1 \times W_2, S_1 \sqcup S_2)$$

where the elements of S_1 commute with those of S_2. A subset $T = T_1 \cup T_2$, $T_i \subset S_i$, is spherical if and only if T_1 and T_2 are both spherical. It follows that

$$\mathcal{S}(W, S) \cong \mathcal{S}(W_1, S_1) \times \mathcal{S}(W_2, S_2)$$

Definition 5.5. A **convex cell complex** is a collection Λ of convex polytopes in an affine space $\mathbb{A}$ such that

1. if $P \in \Lambda$ and F is a face of P, then $F \in \Lambda$ and
2. for any two polytopes P and Q in Λ, either $P \cap Q = \emptyset$ or $P \cap Q$ is a common face of both polytopes.

The elements of Λ are called **cells**. A subset Λ' of Λ is a **subcomplex** if it satisfies (1). If each cell of Λ is a simplex, then Λ is a **simplicial complex**. The **underlying space** of Λ is, as a set, given by

$$X(\Lambda) := \bigcup_{P \in \Lambda} P.$$

If Λ is locally finite (that is, each cell in Λ is a face of only finitely many other cells in Λ), then $X(\Lambda)$ is given the induced topology as a subspace of $\mathbb{A}$. Otherwise, it is topologized as the direct limit of the underlying spaces of its finite subcomplexes.

Definition 5.6. A space X is a **polyhedron** if it is homeomorphic to (the underlying space of) a convex cell complex.

Example 5.7. Given a convex polytope P, the set of all its nonempty faces is a convex cell complex (also denoted as P).

The boundary complex ∂P consists of all proper faces. If P is n-dimensional, then the underlying space of P is an n-disk and the underlying space of ∂P is an $(n-1)$-sphere.

Definition 5.8. The **standard n-simplex** Δ^n is the convex hull of the standard basis $e_1, \ldots, e_{n+1}$ in $\mathbb{R}^{n+1}$, that is,

$$\Delta^n = \left\{ \sum_{i=1}^{n+1} \lambda_i e_i \mid \lambda_i \geq 0,\ \sum_{i=1}^{n+1} \lambda_i = 1 \right\}$$

of the standard basis $e_1, \ldots, e_{n+1}$ in $\mathbb{R}^{n+1}$.

Let X be an abstract simplicial complex with vertex set V. We associate to X a convex cell complex $\mathrm{Geom}(X)$ by identifying each n-simplex Δ in X with the standard n-simplex Δ^n; this gives the n-cells in the associated simplicial cell complex. We call $\mathrm{Geom}(X)$ the **standard geometric realisation** of X.

Definition 5.9. Let $\mathcal{P}$ be a partially ordered set. A **chain** is a totally ordered subset of $\mathcal{P}$. The **flag complex** (or **order complex**) $\mathrm{Flag}(\mathcal{P})$ of $\mathcal{P}$ is the abstract simplicial complex of all finite chains in $\mathcal{P}$.

The **geometric realization of a poset** $\mathcal{P}$ is the geometric realization of $\mathrm{Flag}(\mathcal{P})$,

$$|\mathcal{P}| := \mathrm{Geom}(\mathrm{Flag}(\mathcal{P})).$$

That is, we map finite chains in $|\mathcal{P}|$ with $(n+1)$ elements to an n-simplex, and the elements of $\mathcal{P}$ are vertices of $|\mathcal{P}|$.

Definition 5.10. Let K denote the geometric realization of the poset $\mathcal{S}$. Let $W\mathcal{S}$ denote the poset

$$W\mathcal{S} := \bigsqcup_{T \in \mathcal{S}} W/W_T = \{wW_T \mid w \in W,\ T \subseteq S \text{ spherical}\}$$

partially ordered by inclusion $A \subseteq B$. (Note that the poset $W\mathcal{S}$ is a disjoint union: by Corollary 3.4, $wW_T = w'W_{T'}$ if and only if $T = T'$ and $w^{-1}w' \in W_{T'}$.)

Definition 5.11. The **Davis complex** Σ is the geometric realization $|W\mathcal{S}|$. The group W acts on the poset $W\mathcal{S}$ by

$$\begin{aligned} W \times W\mathcal{S} &\longrightarrow W\mathcal{S}, \\ (w, w'W_T) &\longmapsto (ww')W_T, \end{aligned}$$

which induces an action of W on the geometric realization $\Sigma = |W\mathcal{S}|$:

$$\begin{aligned} W \times |W\mathcal{S}| &\longrightarrow |W\mathcal{S}|, \\ \left(w, \sum_{T_i \subseteq T} \lambda_i (w'W_{T_i}) \right) &\longmapsto \sum_{T_i \subseteq T} \lambda_i (ww'W_{T_i}). \end{aligned}$$

Remark 5.12. The projection $W\mathcal{S} \to \mathcal{S}$ given by $wW_T \to T$ induces a simplicial projection $\pi : \Sigma \to K$. (Notice that by Corollary 3.4, the map $W\mathcal{S} \to \mathcal{S}$ is well-defined.)

Similarly, the inclusion $S \hookrightarrow W\mathcal{S}$ given by $T \to W_T$ induces an inclusion $\imath : K \hookrightarrow \Sigma$. We identify K with its image $\imath(K)$, and call it (as well as any of its translates by an element of W) a **chamber** of Σ.

Proposition 5.13. *Let (W, S) be a Coxeter system with Davis complex $\Sigma(W, S)$. Then the action of W on Σ is proper and cocompact.*

Proof. By definition, Σ is locally finite and given the induced topology as a subspace of $\mathbb{R}^n$. Consider the orbit space

$$W \backslash \Sigma = \{[W_T] \mid T \subseteq S \text{ spherical}\}.$$

$W \backslash \Sigma$ is homeomorphic to the chamber K (compare [Davis, p.64], [Davis, Theorem 7.2.4]). Because S is finite, K is compact. It follows that $W \backslash \Sigma$ is compact, that is, W acts cocompactly on Σ. The remaining properties are clear. $\square$

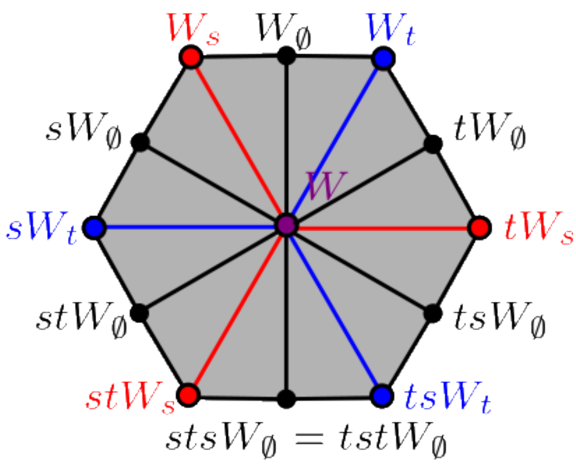


Figure 13: Davis-Komplex for D_6 (as poset)

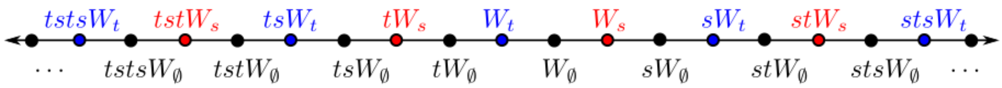


Figure 14: Davis-Komplex for D_∞ (as poset)

5.2 The Davis complex as CW complex

We now wish to endow the Davis complex Σ with a cell structure, coarser than its simplicial structure, such that each cell is a Coxeter polytope.

Definition 5.14. A **CW complex** is a filtration

$$\emptyset = X^{(-1)} = X^{(0)} \subset X^{(1)} \subset \cdots \subset X^{(n)} = X, \qquad n \in \mathbb{N}$$

which is defined inductively over:

- $X^{(0)} = \bigsqcup\{\mathrm{Pt}\}$, equipped with the discrete topology;
- $X^{(j)} := {}^{X^{(j-1)} \bigsqcup_\alpha D^j_\alpha}\big/_{x \sim f_\alpha(x)}$ with the quotient topology defined by the **attaching map**

$$f_\alpha : \delta D^j_\alpha \to X^{(j-1)},$$

where D^j are j-dimensional balls. $X^{(j)}$ is called the j-**skeleton**.

The following Lemma shows that when a Coxeter group W is finite, we can identify the simplicial complex $\Sigma(W, S)$ with the barycentric subdivision of the associated Coxeter polytope. That is, $\Sigma(W, S)$ is topologically a cell.

Lemma 5.15 ([Davis, Lemma 7.3.3])**.** *Suppose W is finite and that C is its associated Coxeter polytope. Let $\mathcal{F}(C)$ be its face poset, and x a generic point. Then the correspondence $w \to wx$ induces an isomorphism of posets, $W\mathcal{S} \cong \mathcal{F}(C)$. In other words, a subset of W corresponds to the vertex set of a face of C if and only if it is a coset of a special subgroup of W.*

In case W is infinite, since the poset $(W\mathcal{S})_{\leq wW_T}$ is isomorphic to $W_T(\mathcal{S}_{\leq T})$, the face corresponding to $\mathcal{S}_{\leq wW_T}$ is isomorphic to the barycentric subdivision of a Coxeter cell of type W_T. So we can put a cell structure on Σ, coarser than its simplicial structure, by identifying each such barycentric subdivision with the corresponding Coxeter cell.

Proposition 5.16 ([Davis, Proposition 7.3.4])**.** *Let (W, S) be a Coxeter system. The **Davis complex** Σ is given inductively by:*

- *0-cells: $\{wW_\emptyset\}$ where $W_\emptyset = \mathbb{1}$. It follows:*

$$\Sigma^{(0)} = \{w \mid w \in W\}$$

- *To construct the 1-skeleton $\Sigma^{(1)}$, consider the cosets:*

$$\{wW_{\{s\}} \mid w \in W,\ s \in S\}, \quad wW_{\{s\}} = \{w, ws\}.$$

  Attach 1-dimensional balls (i.e. intervals) to each pair (of 0-cells) $\{w, ws\}$. As S generates W, it follows that $\Sigma^{(1)}$ is the (geometric realisation of) $\mathrm{Cay}(W, S)$*.*

- *To construct the n-skeleton $\Sigma^{(n)}$, consider the cosets:*

$$U := \{wW_T \mid w \in W,\ T \subseteq S \text{ spherical},\ |T| = n\}.$$

  Attaching n-dimensional balls to $u \in U$ in $\Sigma^{(n-1)}$ then results in the n-skeleton $\Sigma^{(n)}$.

Remark. Since the generating set S of a Coxeter system (W, S) is finite, we have $\Sigma = \Sigma^{(n)}$ for some $n < \infty$. In general, Σ may however have infinitely many cells, for example when W is infinite.

The Davis complex is simply connected, as stated by the following proposition.

Proposition 5.17 ([Thomas, Lemma 5.26], [Davis, 7.3.5])**.** *Σ is simply connected, i.e. Σ is path-connected and $\pi_1(\Sigma)$ is the trivial group.*

Example 5.18. Let $W \cong D_6$. The 0-skeleton $\Sigma^{(0)}$ is given by $\{1, s, st, sts, ts, t\}$. For every coset $wW_{\{s\}}$ and $wW_{\{t\}}$ we have the intervals:

$\{\mathbb{1}, s\}$	
$\{s, st\}$	
$\{st, sts\}$	
$\{sts, ts\}$	
$\{t, ts\}$	
$\{1, t\}$	

Table 1: 1-cells for D_6

We attach these intervals to the corresponding vertices. In the 2-skeleton we have a disc D^2, attached to the vertices $W_{\{s,t\}} = W$. ("Fill the Cayley graph")

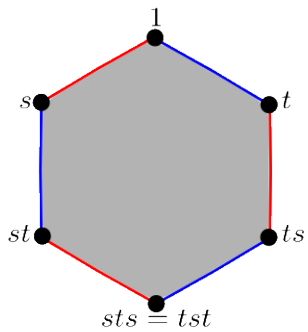


Figure 15: Davis complex for D_6 (as CW-Komplex)

In general, if W is finite, there is an $|S|$-dimensional cell, $|W|$ 0-dimensional cells, and $|T|$-dimensional sells for all subsets $T \subseteq S$.

5.3 The CAT(0) inequality

The definition of Σ as a CW complex has allowed us to easily derive its *topological* properties. We however have little information on the attaching maps. We now wish to define a metric on the Davis complex Σ, such that this metric satisfies the *CAT(0) inequality.*

Definition 5.19. A **triangle** Δ in a metric space X is a configuration of three geodesic segments ("edges") connecting three points ("vertices") in pairs.

A **(Euclidean) comparison triangle** for Δ is a triangle Δ^* in $\mathbb{R}^2$ with the same edge lengths. (Such comparison triangles always exist.)

If Δ^* is a comparison triangle for Δ, then for each edge of Δ there is a well-defined isometry, denoted by $x \to x^*$, which takes the given edge of Δ onto the corresponding edge of Δ^*.

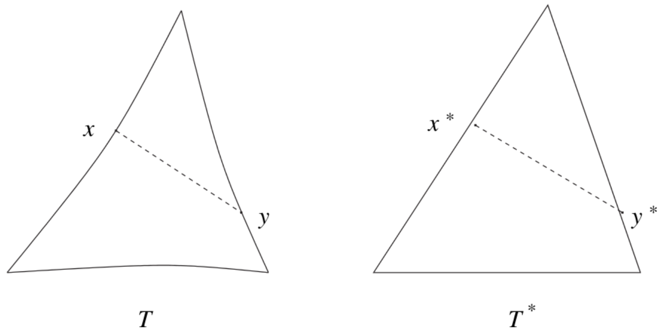


Figure 16: The CAT(0)-inequality

Definition 5.20. A metric space X **satisfies the CAT(0) inequality** (or is a **CAT(0)-space**) if the following two conditions hold:

1. X is a geodesic space;

2. For any triangle Δ, and any two points $x, y \in \Delta$, we have:

$$d(x, y) \leq d^*(x^*, y^*)$$

where x^*, y^* are the corresponding points in the comparison triangle Δ^*, and d^* is the distance in $\mathbb{R}^2$.

Remark 5.21. Similarly, a geodesic metric space X is CAT(-1) if geodesic triangles in X are “no fatter” than comparison triangles in $\mathbb{H}^2$.

A metric space X is CAT(1) if all points in X at distance $< \pi$ are connected by geodesics, and all geodesic triangles in X with perimeter $< 2\pi$ are “no fatter” than comparison triangles in a hemisphere of $\mathbb{S}^2$.

Example 5.22.

- Pre-Hilbert spaces are CAT(0).

- When endowed with the induced metric, a convex subset of Euclidean space $\mathbb{R}^n$ is CAT(0).

- Hyperbolic space $\mathbb{H}^n$ is CAT(0). Generally, we can show that CAT(-1) spaces are CAT(0) and CAT(1).

5.4 The Davis complex is CAT(0)

We will add a metric to Σ as follows. First we will define a *Coxeter polytope* through the Tits representation of (W, S). We will then assign a (fixed) polytope of this type to every cell in Σ, turning Σ into a *polyhedral complex.*

Definition 5.23. A (euclidean) **polyhedral complex** is a (finite-dimensional) CW complex, where every n-cell is metrised as a convex polytope in $\mathbb{R}^n$, and the restrictions of the attaching maps to codimension-1 faces are isometries.

Theorem 5.24 ([Bridson, I.7.19])**.** *If a connected polyhedral complex X has finitely many isometry types of cells, then X is a complete geodesic space.*

We wish to show:

Theorem 5.25. *Σ is a complete CAT(0) space.*

Proof. We show how to endow Σ with a piecewise Euclidean metric. Choose a sequence of positive real numbers $\underline{d} = (d_s)_{s\in S}$ with each $d_s > 0$. For every finite W_T, let

$$\sigma_T : W_T \to GL_n(\mathbb{R}), \quad n = |T|$$

denote the Tits representation. For every $t \in T$, the reflection σ_t fixes the hyperplane H_t with unit normal vector e_t, and for $t, t' \in T$, the hyperplanes H_t, $H_{t'}$ meet at dihedral angle $\frac{\pi}{m}$, where $\langle t, t'\rangle \cong D_{2m}$. Let C_T be the chamber given by

$$C_T = \{x \in \mathbb{R}^n \mid B(x, e_t) \geq 0 \ \forall t \in T\}.$$

Then there is a unique $x_T = x_T(\underline{d})$ such that $d(x_T, H_t) = d_t > 0$ for any $t \in T$. We now identify every cell of Σ with vertex set wW_t with the Coxeter polytope of C_T, i.e. the convex hull of the W_T orbit of x_T.

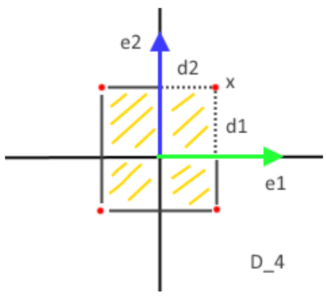


Figure 17: Coxeter polytope for the action of $W = D_4$

If we set $d_s = \frac{1}{2}$ for all $s \in S$, then every edge in the 1-skeleton has length 1. This implies:

- Σ is a polyhedral complex.
- There are only finitely many isometry classes of cells.

By Theorem 5.24, Σ is then a complete geodesic space. Furthermore, Σ is simply connected by Proposition 5.17. (In particular, Σ equals its universal cover $\tilde{\Sigma}$.) By the **Cartan-Hadamard theorem for CAT(0) spaces**, it then suffices to show that Σ is *locally* CAT(0). For details, see [Davis, Section 12.1].

$\square$

Theorem 5.26 (Cartan-Hadamard theorem for CAT(0) spaces, [Bridson, II.4.1])**.** *Let X be a complete, connected geodesic metric space. If X is locally CAT(0), then the universal cover of X is CAT(0).*

Remark 5.27. In general polyhedral complexes are not CAT(0). Let X denote the 2-skeleton of a cube in $\mathbb{R}^3$. Geodesic triangles in X which contain a vertex x are "thicker" than comparison triangles in $\mathbb{R}^2$.

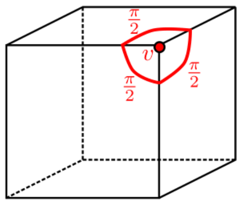


Figure 18: A polyhedral complex which is not CAT(0).

6 Flats in the Davis complex

In general, the Davis complex is not a manifold. As with symmetric spaces, we study flat subspaces (flats) in the Davis complex, that is, spaces which are isometric to $\mathbb{R}^n$ for some $n \in \mathbb{N}$. In particular, we are interested in collections of flats with *isolated* elements. We recall a few basic definitions in CAT(0) geometry before expanding on this statement.

Definition 6.1. A subset Y of a CAT(0) space (X, d) is called **convex** if the geodesic segment joining any two points of Y is entirely contained in Y. A map $f : X \to \mathbb{R}$ is a **convex map** if for each geodesic $\rho : I \to X$, the composed map $f \circ \rho : I \to \mathbb{R}$ is convex. In that case, sublevel sets of f are convex subsets of X.

Clearly, a convex subset of a CAT(0) space is itself a CAT(0) space when endowed with the induced metric.

Lemma 6.2 ([Bridson, Cor. II. 2.5]). *Given a complete convex subset $Y \subset X$, the distance to Y, namely*

$$\begin{aligned} d_Y : X &\longrightarrow \mathbb{R} \\ x &\longmapsto d(x, Y) = \inf_{y \in Y} d(x, y) \end{aligned}$$

*is a convex map. Its sublevel sets are called **tubular neighborhoods** of Y and denoted by $\mathcal{N}_r(Y) = d_Y^{-1}([0, r])$.*

6.1 Products in the Davis complex

Recall that a group **virtually** has some property if a subgroup of finite index has the property. For example, a finite group is virtually trivial. Since Coxeter groups have faithful linear representations, they are virtually torsion free. [Davis, Corollary D.1.4] In this section, we characterize when Coxeter groups are virtually abelian, and the consequences for the Davis complex.

Theorem 6.3 ([Davis, Theorem 12.3.5], [Vinberg]). *Let (W, S) be an irreducible Coxeter system. Then W is virtually abelian if and only if W is either a finite or an affine Coxeter group.*

Corollary 6.4. *Let (W, S) be a Coxeter system, and $T \subset S$ a subset such that W_T is virtually abelian. Then $\Sigma(W_T, T)$ is a product of $\mathbb{E}^n$ (for some $n \in \mathbb{N}$) and compact polyhedra P.*

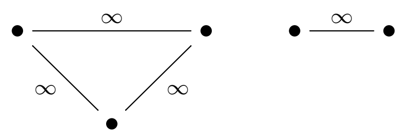
Proof.

- Let W be finite. Then by definition, the Davis complex Σ is a compact polyhedron.

- Let W be an affine Coxeter group. Then Σ is a tesselation of $\mathbb{E}^n$ by a simple convex polytope P.
- Suppose that W decomposes as $(W, S) = (W_1 \times W_2, S_1 \sqcup S_2)$. Then $\mathcal{S}(W, S) = \mathcal{S}(W_1, W_2) \times \mathcal{S}(W_2, S_2)$ by Example 5.4, and a Coxeter polytope C_{W_T} decomposes as $C_{W_{T_1}} \times C_{W_{T_2}}$ by Example 2.5.3. It follows that $\Sigma(W, S)$ decomposes as $\Sigma(W_1, S_1) \times \Sigma(W_2, S_2)$.
- By Theorem 6.3, W_T is the product of affine and finite Coxeter groups. By the above, the claim then holds.

$\square$

Example 6.5. Consider $W := (\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2) \times (\mathbb{Z}_2 * \mathbb{Z}_2)$ with generating set $S := \{s_1, s_2, s_3\} \sqcup \{t_1, t_2\}$ and Coxeter diagram:



The Cayley graph is given by the Cartesian product of the Cayley graphs of the direct factors C_2^{*3} and C_2^{*2}. Thus we know the 1-skeleton. Furthermore, as $|S| = 5$ and W is an infinite group, we have at most spherical subgroups of rank 4. We consider the different cases:

Case 1. $|T| = 2$, that is

$$\begin{aligned} T = &\{s_1, t_1\} \vee \{s_1, t_2\} \vee \{s_1, s_2\} \vee \{s_1, s_3\} \vee \{s_2, s_3\} \\ &\vee \{t_1, t_2\} \vee \{s_2, t_1\} \vee \{s_2, t_2\} \vee \{s_3, t_1\} \vee \{s_3, t_2\} \end{aligned}$$

$W_{\{s_1,s_2\}}$, $W_{\{t_1,t_2\}}$, $W_{\{s_2,s_3\}}$ and $W_{\{s_1,s_3\}}$ are infinite dihedral groups. This leaves the groups $W_{\{s_i,t_j\}}$, $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$. The s_i and t_j commute by definition, thus the corresponding groups are finite:

$$\begin{aligned} \langle s_1, t_1 \rangle &= \{1, s_1, t_1, s_1t_1\}, & \langle s_1, t_2 \rangle &= \{1, s_1, t_2, s_1t_2\}, \\ \langle s_2, t_1 \rangle &= \{1, s_2, t_1, s_2t_1\}, & \langle s_2, t_2 \rangle &= \{1, s_2, t_2, s_2t_2\}, \\ \langle s_3, t_1 \rangle &= \{1, s_3, t_1, s_3t_1\}, & \langle s_3, t_2 \rangle &= \{1, s_3, t_2, s_3t_2\}. \end{aligned}$$

These are precisely the (vertex sets of the) cells of the "filled in" Cayley graph.

Case 2. $|T| = 3$, then either W_T must contain an infinite dihedral subgroup, or $W_T = \langle s_1, s_2, s_3 \rangle$ which is infinite. The case $|T| = 4$ is similar.

6.2 Flat subspaces

Definition 6.6. Let (W, S) be a Coxeter system. A **flat subspace** in the Davis complex Σ is a subset $F \subseteq \Sigma$ which is isometric to $\mathbb{R}^n$ for some $n \geq 2$. We call a flat subspace **special** if there exists a special subgroup W_T for $T \subseteq S$ such that $\langle J \rangle$ is virtually abelian.

Definition 6.7. Let $\mathfrak{F}$ be a collection of closed convex subsets of Σ. We say the elements of $\mathfrak{F}$ are **isolated** in Σ if the following conditions:

(A) There is a constant $D < \infty$ such that each flat F of Σ lies in a D-tubular neighborhood of some $C \in \mathfrak{F}$.

(B) For each positive $r < \infty$ there is a constant $\rho = \rho(r) < \infty$ so that for any two distinct elements $C, C' \in \mathfrak{F}$ we have

$$\operatorname{diam}(\mathcal{N}_r(C) \cap \mathcal{N}_r(C')) < \rho,$$

where $\mathcal{N}_r(C)$ denotes the r-tubular neighborhood of C.

If $\mathfrak{F}$ consists of flats, we say that Σ has **isolated flats**.

The main result for isolated flats is given by the following theorem.

Proposition 6.8. *Let (W, S) be a Coxeter system. The following assertions are equivalent:*

1. *For all non-spherical $J_1, J_2 \subset S$ such that $[J_1, J_2] = 1$, the group $\langle J_1 \cup J_2 \rangle$ is virtually abelian.*
2. *For each minimal hyperbolic $J \subset S$, the set $J^\perp$ is spherical.*

Proof. We show the equivalence $\neg(i) \Leftrightarrow \neg(ii)$.

- $\neg(i) \Rightarrow \neg(ii)$.

  Let $J_1, J_2 \subset S$ such that $\langle J_1 \cup J_2 \rangle$ is not virtually abelian. Note that $[J_1, J_2] = 1$ implies that $J_2 \subset J_1^\perp$ (*), and that J is the direct product $\langle J_1 \rangle \times \langle J_2 \rangle$. Then either J_1 or J_2 is non-affine and non-spherical by Theorem 6.3. Denote this set by J.

  By assumption both J_1 and J_2 are non-spherical, thus by (*) $J^\perp$ is non-spherical as well. Any *minimal* non-spherical and non-affine subset I of J is minimal hyperbolic, and since $I \subset J$ we have $I^\perp \supset J^\perp$. In particular, $I^\perp$ is non-spherical, failing (ii).

- $\neg(ii) \Rightarrow \neg(i)$.

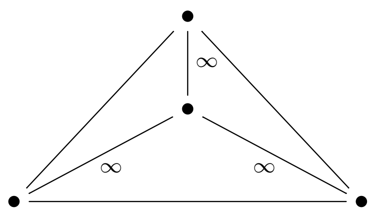
  Assume there is some minimal hyperbolic $J \subset S$ such that $J^\perp$ is non-spherical. Then J is non-spherical, $[J, J^\perp] = 1$ and the group $\langle J \cup J^\perp \rangle$ is the direct product of a non-spherical, non-affine Coxeter group with a non-spherical Coxeter group, failing (i).

□

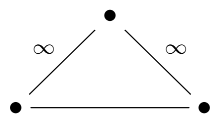
Theorem 6.9 ([Caprace, Corollary D]). *Let (W, S) be a Coxeter system. The Davis complex Σ has isolated flats if and only the assertions in Proposition 6.8 are satisfied.*

Using the above criterion, we give an example of a Coxeter system (W, S) where the Davis complex *has isolated flats.*

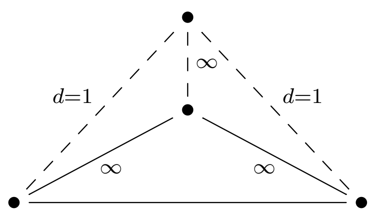
Example 6.10. Let $W := \mathbb{Z}_2 * (3, 3, 3)$ with Coxeter diagram



We show that W satisfies the conditions of Theorem 6.9. First note that (W, S) is *not* minimally hyperbolic – none of the possible diagrams for minimally hyperbolic groups of rank 4 match. We thus look at rank ≤ 3 subsets $J \subset S$ which may be. (Note there are no hyperbolic groups of rank ≤ 2. In particular $\bullet\!\!-\!\!-\!\!\bullet$ and $\bullet\overset{\infty}{-\!\!-}\bullet$ are dihedral groups, resp. spherical and affine.) By symmetry, it suffices to consider the subdiagram:



The set $J^\perp$ is given by all vertices with distance $d > 1$ to J:



It follows that $J^\perp = \emptyset$, thus $W_{J^\perp} = \{1\}$. In particular, $J^\perp$ is spherical. By the theorem, the Davis complex $\Sigma(W, S)$ has isolated flats.

We now give an example of a Coxeter group where the Davis complex Σ does *not* have isolated flats.

Example 6.11. Let $W' := (\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2) \times (\mathbb{Z}_2 * \mathbb{Z}_2)$ be the Coxeter group from Example 6.5. We first compute which special subgroups result in flat subspaces in Σ.

Case 1. $|T| = 3$. The "ladders" $\langle s_i, t_1, t_2\rangle$, $i \in \{1, 2, 3\}$ are homeomorphic to $\mathbb{R} \times [0, 1]$ and equipped with a piecewise Euclidean metric. As $[0, 1]$ is compact, these spaces cannot be homeomorphic (in particular, not isometric) to $\mathbb{R}^2$. The cases $\langle s_i, s_j, t_k\rangle$ with $k \in \{1, 2\}$ are similar.

Case 2. $|T| = 4$. The groups $\langle s_i, s_j, t_1, t_2\rangle$ are isometric to $\mathbb{R}^2$, and thus result in a flat subspace in $\Sigma(W, S)$. Note that $\langle s_1, s_2, s_3, t_k\rangle$ are homeomorphic to $[0, 1] \times \mathbb{R}^2$ and thus not isometric to $\mathbb{R}^3$.

Now assume that Σ has isolated flats. In particular, there is a collection of flats $\mathfrak{F}$ which satisfies condition (B). Let $r = 1$ and consider the tubular neighborhoods $\mathcal{N}_1(\Sigma(W_1, S_1))$ and $\mathcal{N}_1(\Sigma(W_2, S_2))$. Then these neighbourhoods intersect in the "ladder" $\Sigma(\langle s_1, t_1, t_2\rangle)$, which has infinite diameter; a contradiction.

We could read directly from the diagram that W' does not have isolated flats, again by 6.9. Let $J_1 := \{s_1, s_2, s_3\}$ and $J_2 := \{t_1, t_2\}$. Then clearly $[J_1, J_2] = 1$ and J_1, J_2 are non-spherical. However, $\langle J_1 \cup J_2\rangle = W'$ is not virtually abelian, as the factor W_{J_1} is not an affine Coxeter system.

References

[Bahls] P. Bahls. *The isomorphism problem in Coxeter groups*, 2005.

[Abramenko] P. Abramenko, K. Brown. *Buildings - Theory and Applications*, 2008.

[Bjorner] A. Bjorner, F. Brenti. *Combinatorics of Coxeter Groups*, 2005.

[Elements] N. Bourbaki. *Éléments de mathématique - Algèbre - Chapitres 1 à 3*, 1981.

[Groupes] N. Bourbaki. *Groupes et algèbres de Lie - Chapitres 4 à 6*, 1981.

[Bridson] M.R. Bridson. *Metric spaces of non-positive curvature*, 1999.

[Caprace] P. Caprace. *Buildings with isolated subspaces and relatively hyperbolic Coxeter groups*, 2013.

[Davis] M.W. Davis. *The geometry and topology of Coxeter groups*, 2008.

[Hatcher] A. Hatcher. *Algebraic Topology*, 2001.

[Hall] B. Hall. *Lie Groups, Lie Algebras, and Representations*, 2003.

[Humphreys] J. E. Humphreys. *Reflection Groups and Coxeter Groups*, 1990.

[Imrich] W. Imrich, S. Klavžar. *Product graphs, Structure and Recognition,* 2000.

[Loeh] C. Löh. *Introduction to geometric group theory*, 2015.

[Ol'shanskii] A. Yu. Ol'shanskii. *Geometry of defining relations in groups*, 1991.

[Massey] W. S. Massey. *A Basic Course in Algebraic Topology*, 1991.

[Suter] R. Suter. *Spiegelungsgruppen*, 2008.

[Thomas] A. Thomas. *Geometric and Topological Aspects of Coxeter Groups and Buildings*, 2018.

[Vinberg] G. Margulis, E. B. Vinberg. *Some linear groups virtually having a free quotient*, Journal of Lie Theory, 10:171-180, 2000.